WAVE FRONT SETS OF REDUCTIVE LIE GROUP REPRESENTATIONS III

BENJAMIN HARRIS AND TOBIAS WEICH

Abstract. In this article, the authors prove an upper bound for the wave front set of an induced Lie group representation under a uniformity condition, which is verified for several large classes of examples. As a corollary, if X is a homogeneous space for a real, reductive algebraic group G with a nonzero invariant density, the authors give a complete description of the regular, semisimple asymptotics of the support of the Plancherel measure for $L^2(X)$.

1. Introduction

This is the third in a series of papers on wave front sets of reductive Lie group representations [HHO],[Har].

If f is a continuous function on a Lie group G, then $\mathrm{WF}_e(f) = \mathrm{WF}(f) \cap iT_e^*G$ (resp. $\mathrm{SS}_e(f) = \mathrm{SS}(f) \cap iT_e^*G$) denotes the wave front set of f (resp. singular spectrum of f) intersected with the fiber of T^*G over the identity. If G is a Lie group and (π, V) is a unitary representation of G, then the *wave front set* and *singular spectrum* of π are defined by

$$\mathrm{WF}(\pi) = \overline{\bigcup_{u,v\in V} \mathrm{WF}_e(\pi(g)u, v)},\ \mathrm{SS}(\pi) = \overline{\bigcup_{u,v\in V} \mathrm{SS}_e(\pi(g)u, v)}.$$

In words, the wave front set of π is the closure of the unions of the wave front sets at the identity of the matrix coefficients of π. These notions were first introduced by Kashiwara-Vergne [KV79] and Howe [How81]. It is a well known fact that $\mathrm{WF}(\pi)$ and $\mathrm{SS}(\pi)$ are $\mathrm{Ad}^*(G)$ invariant, closed cones in $i\mathfrak{g}^* = iT_e^*G$. For precise definitions of the wave front set and singular spectrum which are given in a way that is compatible with the exposition in this paper see Section 2 of the first paper in this series [HHO].

Let G be a Lie group, let $H \subset G$ be a closed subgroup, and let (τ, V_τ) be a unitary representation of H. Let $X = G/H$, let $\mathcal{D}^{1/2} \to X$ denote the bundle of complex half densities on X (see Appendix A), and let $\mathcal{V}_\tau = G \times_H V_\tau \to X$ denote the unique G equivariant vector bundle on X whose fiber over $\{H\}$ is the unitary representation τ. Then the representation of G *induced* from the representation (τ, V_τ) of H is defined by

$$\mathrm{Ind}_H^G \tau := L^2(X, \mathcal{D}^{1/2} \otimes \mathcal{V}_\tau)$$

with the natural action of G on the square integrable sections on the vector bundle $\mathcal{D}^{1/2} \otimes \mathcal{V}_\tau \to X$.

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In this article, we are interested in relating $\mathrm{WF}(\tau)$ and $\mathrm{WF}(\mathrm{Ind}_H^G(\tau))$ (resp. $\mathrm{SS}(\tau)$ and $\mathrm{SS}(\mathrm{Ind}_H^G(\tau))$). Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) denote the Lie algebra of G (resp. H), let $i\mathfrak{g}^*$ (resp. $i\mathfrak{h}^*$) be the set of purely imaginary linear functionals on $\mathfrak{g}$ (resp. $\mathfrak{h}$). Then recall that $\mathrm{WF}(\tau), \mathrm{SS}(\tau) \subset i\mathfrak{h}^*$ and $\mathrm{WF}(\mathrm{Ind}_H^G(\tau)), \mathrm{SS}(\mathrm{Ind}_H^G(\tau)) \subset i\mathfrak{g}^*$ are closed, $\mathrm{Ad}^*(G)$ invariant cones. Using the natural projection $q\colon i\mathfrak{g}^* \to i\mathfrak{h}^*$ we define a natural way of inducing closed $\mathrm{Ad}^*(G)$ invariant cones in $i\mathfrak{g}^*$ from closed, $\mathrm{Ad}^*(H)$ invariant cones in $i\mathfrak{h}^*$: If $S \subset i\mathfrak{h}^*$ is a subset of $i\mathfrak{h}^*$, then we may form the *induced* subset of $i\mathfrak{g}^*$ by

$$\mathrm{Ind}_H^G S := \overline{\mathrm{Ad}^*(G) \cdot q^{-1}(S)}.$$

By construction, this is a closed $\mathrm{Ad}^*(G)$ invariant cone in $i\mathfrak{g}^*$ if S is a closed, $\mathrm{Ad}^*(H)$ invariant cone in $i\mathfrak{h}^*$.

In Theorem 1.1 of [HHO], it is shown that

$$\mathrm{WF}(\mathrm{Ind}_H^G \tau) \supset \mathrm{Ind}_H^G \mathrm{WF}(\tau)$$

and

$$\mathrm{SS}(\mathrm{Ind}_H^G \tau) \supset \mathrm{Ind}_H^G \mathrm{SS}(\tau).$$

In this paper, we address the converse statement. First, we address the case where $X = G/H$ has an invariant measure and $\tau = \mathbb{1}$ is the trivial representation of H.

Theorem 1.1. *Suppose $X = G/H$ is a homogeneous space for a Lie group G equipped with a nonzero invariant density. Then*

$$\mathrm{WF}(L^2(X)) = \mathrm{WF}(\mathrm{Ind}_H^G \mathbb{1}) = \mathrm{Ind}_H^G \mathrm{WF}(\mathbb{1}) = \overline{\mathrm{Ad}^*(G) \cdot (\mathfrak{g}/\mathfrak{h})^*}$$

and

$$\mathrm{SS}(L^2(X)) = \mathrm{SS}(\mathrm{Ind}_H^G \mathbb{1}) = \mathrm{Ind}_H^G \mathrm{SS}(\mathbb{1}) = \overline{\mathrm{Ad}^*(G) \cdot (\mathfrak{g}/\mathfrak{h})^*}.$$

This result is proven in Section 2. Next, we address the case where τ is a finite dimensional, unitary representation of H.

Theorem 1.2. *Suppose G is a real, linear algebraic group, suppose $H \subset G$ is a closed subgroup with Lie algebra $\mathfrak{h}$, and suppose τ is a finite dimensional, unitary representation of H. Assume the existence of a closed, real algebraic subgroup $H_1 \subset G$ with Lie algebra $\mathfrak{h}$ and assume that the set of semisimple elements in $\mathfrak{h}$ is dense in $\mathfrak{h}$. If τ is a finite dimensional, unitary representation of H, then*

$$\mathrm{WF}(\mathrm{Ind}_H^G \tau) = \mathrm{Ind}_H^G \mathrm{WF}(\tau)$$

and

$$\mathrm{SS}(\mathrm{Ind}_H^G \tau) = \mathrm{Ind}_H^G \mathrm{SS}(\tau).$$

Observe that the hypotheses of Theorem 1.2 hold whenever H is a real, reductive algebraic group or a parabolic subgroup of a real, reductive algebraic group. However, the hypotheses fail when H is a unipotent group. We want to emphasize that the assumption of dense semisimple elements is not only of technical nature in our proof, but that one cannot hope to prove the upper bounds on the wavefront sets without further assumptions. Already for the classical example of $G = \mathrm{SL}(2, \mathbb{R})$ and $H = N$ being the standard unipotent subgroup (thus not fulfilling the dense semisimple condition) one finds counterexamples to the equality $\mathrm{WF}(\mathrm{Ind}_H^G \tau) = \mathrm{Ind}_H^G \mathrm{WF}(\tau)$. A family of such counterexamples is presented in

Section 7 where we combine unpublished work of Harish-Chandra (later discovered independently by Wallach [Wal92]) with Matumoto's work on the theory of Whittaker functionals for real, reductive algebraic groups [Mat92].

Next, we consider the case where τ is an arbitrary, unitary representation of H and $X = G/H$ is compact.

Theorem 1.3. *Suppose G is a Lie group, $H \subset G$ is a closed subgroup, $X = G/H$ is a compact homogeneous space, and τ is a unitary representation of H. Then*

$$\mathrm{WF}(\operatorname{Ind}_H^G \tau) = \operatorname{Ind}_H^G \mathrm{WF}(\tau)$$

and

$$\mathrm{SS}(\operatorname{Ind}_H^G \tau) = \operatorname{Ind}_H^G \mathrm{SS}(\tau).$$

This result was obtained by Kashiwara-Vergne in the case where G is compact [KV79]. Suppose G is a real, reductive algebraic group, $H = P \subset G$ is a parabolic subgroup, and $P = MAN$ the Langlands decomposition of G. When combined with work of Rossmann [Ros95], the work of Barbasch and Vogan implies Theorem 1.3 in the special case where τ is an irreducible, unitary representation of MA extended to P [BV80]. Although we do not give applications of Theorem 1.3 in this paper, the authors believe that the case where τ is a highly reducible representation of MA will likely play an important role in future work.

Let us now turn to applications of Theorem 1.1 and Theorem 1.2. We require some notation. If G is a real, reductive algebraic group, we denote by $\widehat{G}$ the set irreducible, unitary representations of G equipped with the Fell topology. Any unitary representation (π, V) of G can be written in an essentially unique way as an integral against a measure on the unitary dual. The support of this measure on $\widehat{G}$ is called the *support* of π and is denoted by $\operatorname{supp}\pi$ or $\operatorname{supp} V$.

Within the space $\widehat{G}$, we have the closed subspace, $\widehat{G}_{\mathrm{temp}}$, of irreducible, tempered representations of G. In addition, we have $\widehat{G}_{\mathrm{temp}}'$, the open, dense subspace of $\widehat{G}_{\mathrm{temp}}$ consisting of irreducible, tempered representations of G with regular infinitesimal character. Following Duflo and Rossmann [Duf70], [Ros78], [Ros80], for every $\sigma \in \widehat{G}_{\mathrm{temp}}$, we associate a finite union of coadjoint orbits

$$\mathcal{O}_\sigma \subset i\mathfrak{g}^*$$

to σ. If $\sigma \in \widehat{G}_{\mathrm{temp}}'$, then $\mathcal{O}_\sigma$ is a single G orbit. A subset of a finite dimensional, real vector space, $\mathcal{C} \subset W$, is a *cone* if $t\mathcal{C} = \mathcal{C}$ for any $t > 0$. In addition, if $S \subset W$ is a subset of a finite dimensional, real vector space, then we define the *asymptotic cone* of S in W to be

$$\mathrm{AC}(S) = \left\{ \xi \in W \,\middle|\, \xi \in \mathcal{C} \text{ an open cone } \implies \mathcal{C} \cap S \text{ is unbounded} \right\} \cup \{0\}.$$

If X is a homogeneous space for the group G and $x \in X$, then $G_x \subset G$ denotes the stabilizer of x and $\mathfrak{g}_x$ denotes the Lie algebra of G_x. We identify iT_x^*X with $i(\mathfrak{g}/\mathfrak{g}_x)^* \subset i\mathfrak{g}^*$ in the obvious way. Finally, we let $(i\mathfrak{g}^*)'$ denote the set of regular, semisimple elements in $i\mathfrak{g}^*$.

Corollary 1.4. *Suppose G is a real, reductive algebraic group, and suppose X is a homogeneous space for G with an invariant measure. Then*

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{\mathrm{temp}}}}\mathcal{O}_\sigma\right)\cap(i\mathfrak{g}^*)'=\overline{\bigcup_{x\in X} iT_x^*X}\cap(i\mathfrak{g}^*)'=\overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}\cap(i\mathfrak{g}^*)'.$$

As we will see in Section 6, this Corollary follows from Theorems 1.1 and 1.2 of [HHO], Theorem 1.1 of [Har], and Theorem 1.1 of this paper. We view this statement as the complete determination of the regular, semisimple asymptotics of the support of the Plancherel measure for $L^2(X)$.

Let us consider a family of examples of Corollary 1.4. Let $G=\mathrm{Sp}(2n,\mathbb{R})$ be the set of invertible linear transformations of $\mathbb{R}^{2n}$ that preserve the standard nondegenerate symplectic form on $\mathbb{R}^{2n}$. For every $0\le m\le n-1$, break up $\mathbb{R}^{2n}\cong\mathbb{R}^{2m}\times\mathbb{R}^{2n-2m}$, and consider the closed subgroup $\mathrm{Sp}(2m,\mathbb{R})\times\mathrm{Sp}(2n-2m,\mathbb{Z})$ where the first term preserves the first factor, $\mathbb{R}^{2m}$, and acts by the identity on the second factor, $\mathbb{R}^{2n-2m}$, and the second term preserves the second factor, $\mathbb{R}^{2n-2m}$, and acts by the identity on the first factor, $\mathbb{R}^{2m}$. It follows from Corollary 1.4 (See Section 6 for details) that whenever $2m\le n$, there exist infinitely many discrete series σ of $\mathrm{Sp}(2n,\mathbb{R})$ such that

$$\mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(\sigma,L^2(\mathrm{Sp}(2n,\mathbb{R})/[\mathrm{Sp}(2m,\mathbb{R})\times\mathrm{Sp}(2n-2m,\mathbb{Z})]))\neq\{0\}.$$

We conjecture that whenever $2m>n$ and σ is a discrete series representation of $\mathrm{Sp}(2n,\mathbb{R})$, we have

$$\mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(\sigma,L^2(\mathrm{Sp}(2n,\mathbb{R})/[\mathrm{Sp}(2m,\mathbb{R})\times\mathrm{Sp}(2n-2m,\mathbb{Z})]))=\{0\}.$$

Unfortunately, the techniques of this paper are insufficient to prove this statement. However, we can show the following weaker statement utilizing Corollary 1.4. We require some notation.

Suppose $T\subset G=\mathrm{Sp}(2n,\mathbb{R})$ is a maximal torus with Lie algebra $\mathfrak{t}\subset\mathfrak{g}=\mathfrak{sp}(2n,\mathbb{R})$, and, using the decomposition $\mathfrak{g}=[\mathfrak{t},\mathfrak{g}]\oplus\mathfrak{t}$, identify $i\mathfrak{t}^*\subset i\mathfrak{g}^*$. Let $W=N_G(T)/T$ be the real Weyl group of T with respect to G. Every discrete series representation σ of $\mathrm{Sp}(2n,\mathbb{R})$ corresponds (via its Harish-Chandra parameter) to a single W orbit $\lambda_\sigma=\mathcal{O}_\sigma\cap i\mathfrak{t}^*$. If $\mathcal{C}\subset\overline{\mathcal{C}}\subset(i\mathfrak{t}^*)'$ is an open cone whose closure is contained in the set of regular semisimple elements of $i\mathfrak{t}^*$ and $2m>n$, then there exist at most finitely many discrete series representations σ of Sp(2n, $\mathbb{R}$) such that $\lambda_\sigma\cap\mathcal{C}\neq\emptyset$ and

$$\mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(\sigma,L^2(\mathrm{Sp}(2n,\mathbb{R})/[\mathrm{Sp}(2m,\mathbb{R})\times\mathrm{Sp}(2n-2m,\mathbb{Z})]))\neq\{0\}.$$

In Section 6, we will write down corollaries of Theorem 1.2, and we will consider applications to bundle valued harmonic analysis on homogeneous spaces for real, reductive algebraic groups.

Note that these corollaries likely hold for an arbitrary reductive Lie group of Harish-Chandra class. We state our corollaries for G a real, reductive algebraic group because arguments in the previous paper [Har] utilize results that have only been written down in that special case.

Let us end this introduction with a short outline of the article. In Section 2 we give a direct proof of Theorem 1.1. In Section 3, we formulate a general *wavefront condition U* under which we can prove

$$\mathrm{WF}(\mathrm{Ind}_H^G \tau) = \mathrm{Ind}_H^G \mathrm{WF}(\tau)$$

and an analogous *singular spectrum condition U* under which we show that

$$\mathrm{SS}(\mathrm{Ind}_H^G \tau) = \mathrm{Ind}_H^G \mathrm{SS}(\tau).$$

In Section 4 and Section 5, we verify wavefront condition U and singular spectrum condition U in special cases thereby deducing Theorem 1.2 and Theorem 1.3. Theorem 1.1 could also be obtained by verifying wavefront condtion U and singular spectrum condition U. We nevertheless wanted to include the direct proof of Theorem 1.1 because it nicely illustrates the central ideas which are exploited in the following sections. We therefore also end Section 2 with a discussion of the challenges which one encounters in generalizing these ideas to nontrivial H-representations τ and how they are handled in the subsequent sections. In Section 6 we present additional results that expand upon Corollary 1.4 and we present concrete examples of this result. Finally in Section 7 we attempt to justify the extra hypotheses in Theorem 1.2 by providing counterexamples to a stronger statement.

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2. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

As in the introduction, let G be a Lie group with Lie algebra $\mathfrak{g}$, let $H \subset G$ be a closed subgroup with Lie algebra $\mathfrak{h}$, and assume $X = G/H$ has a nonzero invariant density. If $\eta_0 \notin \overline{\mathrm{Ad}^*(G) \cdot i(\mathfrak{g}/\mathfrak{h})^*}$, then it suffices to show that $\eta_0 \notin \mathrm{SS}(\mathrm{Ind}_H^G \mathbb{1})$. This suffices for the wavefront case as well since $\mathrm{WF}(\mathrm{Ind}_H^G \mathbb{1}) \subset \mathrm{SS}(\mathrm{Ind}_H^G \mathbb{1})$.

To handle the singular spectrum, we require some notation. Choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$, and for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, define the differential operator

$$D^\alpha = \partial_{X_1}^{\alpha_1} \partial_{X_2}^{\alpha_2} \cdots \partial_{X_n}^{\alpha_n}$$

on $\mathfrak{g}$. If $0 \in U_1 \subset U_2 \subset \mathfrak{g}$ are precompact, open sets with $\overline{U_1} \subset U_2$, then (see pages 25-26, 282 of [Hor83]), we may find a sequence of functions $\{\varphi_{N,U_1,U_2}\}$ indexed by $N \in \mathbb{N}$ satisfying the following properties:

- $\varphi_{N,U_1,U_2} \in C_c^\infty(U_2)$

- $\varphi_{N,U_1,U_2}(x) = 1$ if $x \in U_1$
- There exist constants $C_\alpha > 0$ for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that
$$|D^{\alpha+\beta}\varphi_{N,U_1,U_2}(x)| \leq C_\alpha^{|\beta|+1}(N+1)^{|\beta|} \tag{2.1}$$
for every $x \in U_1$ and every multi-index $\beta = (\beta_1, \ldots, \beta_n)$. Here $|\beta| = \beta_1 + \cdots + \beta_n$.

For every pair of subsets $0 \in U_1 \subset U_2 \subset \mathfrak{g}$, we fix such a sequence of functions $\{\varphi_{N,U_1,U_2}\}$.

The strategy of the proof is as follows. We have to show (see for instance Definition 2.3 of [HHO]) that for fixed $f_1, f_2 \in L^2(X)$, there exists an open neighborhood $\eta_0 \in V_0$ and open subsets $0 \in U_1 \Subset U_2 \subset \mathfrak{g}$ such that
$$(\star) := t^N \left| \int\limits_G \int\limits_X f_1(g^{-1}x) f_2(x) (\log^* \varphi_{N,U_1,U_2})(g) e^{t\langle \xi, \log(g)\rangle} dx dg \right| \leq C^{N+1}(N+1)^N \tag{2.2}$$
uniformly for $t > 0$, $\xi \in V_0$ and for all $N \in \mathbb{N}$ (Note that the constant C must not depend on N). Here dg is a nonzero invariant density on G and dx is a nonzero invariant density on X; note that we must choose $U_2 \subset \mathfrak{g}$ sufficiently small for the logarithm function to be well defined. As φ_{N,U_1,U_2} is compactly supported all the integrals are absolutely convergent and we can interchange the order of integration and get
$$(\star) = t^N \left| \int\limits_X \int\limits_G f_1(g^{-1}x) f_2(x) (\log^* \varphi_{N,U_1,U_2})(g) e^{\langle \xi, \log(g)\rangle} dg dx \right| \tag{2.3}$$
The idea is now to prove the bound (2.2) by integrating the G-integral by parts with respect to a right invariant vector field Y_x on G that depends continuously on the point $x \in X$.

We will now construct these vector fields. First note that
$$\overline{\mathrm{Ad}^*(G) \cdot i(\mathfrak{g}/\mathfrak{h})^*} = \overline{\bigcup_{x \in X} (i\mathfrak{g}/\mathfrak{g}_x)^*}$$
where $\mathfrak{g}_x$ denotes the Lie algebra of the stabilizer subgroup $G_x \subset G$ of the point $x \in X$. We fix an arbitrary, not necessarily $\mathrm{Ad}(G)$ invariant scalar product on $\mathfrak{g}^*$. By this scalar product we can identify $\mathfrak{g} \cong \mathfrak{g}^*$ and obtain
$$(\mathfrak{g}/\mathfrak{g}_x)^* \cong \mathfrak{g}_x^\perp$$
where $\mathfrak{g}_x^\perp$ denotes the orthogonal complement with respect to the chosen scalar product.

For $\eta_0 \notin \overline{\bigcup_{x \in X} i\mathfrak{g}_x^\perp}$, we define a continuous family of normalized elements in the Lie algebra $\mathfrak{g}$ parametrized by $x \in X$,
$$Y_x := \frac{\mathrm{pr}_{\mathfrak{g}_x}(-i\eta_0)}{|\mathrm{pr}_{\mathfrak{g}_x}(\eta_0)|} \in \mathfrak{g}.$$
Here $\mathrm{pr}_{\mathfrak{g}_x}$ denotes the orthogonal projection on the subspace $\mathfrak{g}_x$. Note that Y_x is well defined and continuous in $x \in X$ as $\eta_0 \notin \overline{\bigcup_{x \in X} i\mathfrak{g}_x^\perp}$ implies that $|\mathrm{pr}_{\mathfrak{g}_x}(\eta_0)|$ is bounded away from zero.

If we consider Y_x as a right invariant vector field on G we can study its action on the smooth function $\exp(t\langle\xi, \log(\cdot)\rangle)$ when restricted to $\tilde{U}_2 := \exp(U_2) \subset G$

$$Y_x e^{t\langle\xi,\log(g)\rangle} = \frac{d}{ds}\Big|_{s=0} e^{t\langle\xi,\log(e^{sY_x}g)\rangle} = t\mu(\xi, Y_x, g)e^{t\langle\xi,\log(g)\rangle}$$

where

$$\mu(\xi, Y, g) := \frac{d}{ds}\Big|_{s=0}\langle\xi, \log(e^{sY}g)\rangle. \tag{2.4}$$

In the above expression, when $X \in \mathfrak{g}$, we are writing $e^X := \exp(X)$ for the image of X in G under the exponential map. We can thus insert the operator $t^{-N}\mu(\xi, Y_x, g)^{-N}(Y_x)^N$ in (2.3) in front of $e^{t\langle\xi,\log(g)\rangle}$ and integrate the G integral by parts:

$$(\star) \quad = t^N \left|\int_X\int_G f_1(g^{-1}x)f_2(x)(\log^*\varphi_{N,U_1,U_2})(g)t^{-N}\mu(\xi, Y_x, g)^{-N}Y_x^N e^{\langle\xi,\log(g)\rangle}dgdx\right|$$

$$(2.5) = \left|\int_X\int_G (\mu(\xi, Y_x, g))^{-N} f_1(g^{-1}x)f_2(x)\left(Y_x^N(\log^*\varphi_{N,U_1,U_2})(g)\right)e^{\langle\xi,\log(g)\rangle}dgdx\right|$$

where we used the fact that $Y_x \in \mathfrak{g}_x$ and thus $Y_x f_1(g^{-1}x) = 0$. Utilizing (2.1), we obtain bounds

$$Y_x^N(\log^*\varphi_{N,U_1,U_2})(g) \leq C^{N+1}(N+1)^N$$

for some constant C independent of $g \in \exp(U_2)$. It thus remains to consider the term $\mu(\xi, Y_x, g)^{-N}$. Note that from the definition of μ and Y_x we have

$$|\mu(\eta_0, Y_x, e)| = |\operatorname{pr}_{\mathfrak{g}_x}(\eta_0)|$$

and as remarked above this quantity is bounded away from zero for $x \in X$ so we have a positive constant

$$c := \inf_{x\in X}\operatorname{pr}_{\mathfrak{g}_x}(\eta_0) > 0$$

As $\mu : i\mathfrak{g}^* \times \mathfrak{g} \times G \to \mathbb{C}$ is a smooth function and as Y_x only takes values in the compact unit sphere in $\mathfrak{g}$ we can choose sufficiently small neighborhoods $V_0 \subset i\mathfrak{g}^*$ of η_0 and $\tilde{U}_2 \subset G$ of e such that

$$|\mu(\xi, Y_x, g)| \geq \frac{c}{2}$$

uniformly in $\xi \in V_0, g \in \tilde{U}_2$ and $x \in X$.

Then we obtain for all $N \in \mathbb{N}$

$$\begin{aligned}(\star) \quad &\leq \quad \frac{c}{2}^N C^{N+1}(N+1)^N \int_X\int_{\exp(U_2)} |f_1(g^{-1}x)f_2(x)|dgdx \\ &\leq \quad \frac{c}{2}^N C^{N+1}(N+1)^N \|f_1\|_{L^2(X)}\|f_2\|_{L^2(X)}\operatorname{vol}(\exp(U_2)) \\ &\leq \quad C_1^N(N+1)^N\end{aligned}$$

uniformly in $t > 0$ and $\xi \in V_0$. We have thus established the bound (2.2) and proven Theorem 1.1.

Note that the crucial point in this proof was first the interchanging of the order of integration on X and G and second the partial integration on G performed for each point $x \in X$ into a direction of the stabilizer subalgebra $\mathfrak{g}_x$ of the point x. Only because this x-dependent choice of our differential operator were we able to obtain in (2.5) that the factor $f_1(g^{-1}x)f_2(x)$ is differentiable (even constant) into

this direction (Note that in other directions not in $\mathfrak{g}_x$ this would not have been the case because f_1 is not smooth but only in $L^2(X)$).

Let us now discuss what difficulties arise if we do not induce from the trivial representation but instead start from a finite dimensional unitary H-representation (τ, V). We thus have to consider L^2-sections in the Hermitian vector bundle $\mathcal{V} = G \times_\tau V$. Let $\langle \bullet, \bullet \rangle_{\mathcal{V}_x}$ denote the scalar product in the fibre $\mathcal{V}_x$. Then, in (2.5), we then have to derive $\langle f_1(g^{-1}x), f_2(x) \rangle_{\mathcal{V}_x}$ into a direction $Y_x \in \mathfrak{g}_x$. As τ is finite dimensional and thus smooth this is still possible, but the derivatives do not vanish anymore. Instead there appear powers of the operator $d\tau_x(Y_x)$ where τ_x is the unitary representation of the stabilizer subgroup G_x on the fibre $\mathcal{V}_x$. While for any $x \in X$ the linear operators $d\tau_x(Y_x)$ are bounded, it is in general false, that this bound is uniform in $x \in X$. However the dense semisimple condition introduced in Section 5 will allow us to obtain these uniform bounds and prove the upper bounds on the wavefront sets. Note that the question of uniform bounds of $d\tau_x(Y_x)$ is not only a technical problem of our proof-strategy but the examples in Section 7 show that these non-uniform bounds may indeed lead to larger wavefront sets.

A similar problem occurs if $X = G/H$ admits no G invariant measure anymore and if one has to tensor the Hermition vector bundle $\mathcal{V}$ with the half density bundle $\mathcal{D}^{1/2}$. In this case partial differentiation does not only create derivatives of τ_x but there occur also derivatives of the one dimensional G_x-representations σ_x on the fibres $\mathcal{D}_x^{1/2}$ of the half density bundle. However, we will see that the dense semisimple condition is also suited to find uniform bounds for these terms, so the additional difficulties coming from the density bundles is of the same class as the one coming from inducing from finite dimensional representations τ.

The reason why the partial integration approach would work for finite dimensional (τ, V) comes from the fact that it has a trivial, empty wavefront set, i.e. all matrix coefficients are smooth. If we, however, try to apply this approach to general infinite dimensional representations (τ, V) we face the problem that $\langle f_1(g^{-1}x), f_2(x) \rangle_{\mathcal{V}_x}$ might not be differentiable anymore in the $\mathfrak{g}_x$-direction. The partial integration approach breaks down. We however still can use the principal idea of the partial integration approach, which is to interchange the X and G-integral and use the x-dependent splitting $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{g}_x^\perp$. This way we can reduce our problem to studying the oscillating integrals belonging to the matrix coefficients of $(\tau_x, \mathcal{V}_x)$ (see the calculation at the beginning of Section 3). Of course the same uniform bound problems as in the finite dimensional case also occurs in a more general formulation involving oscillating integrals. In Section 3 we prove the upper bounds on the wavefront set (resp. singular spectrum) under some precise uniformity condition on these oscillating integrals which we term wave front condition U (resp. singular spectrum condition U). The advantage of this approach is that for compact X and arbitrary unitary τ these uniformity conditions can be verified without too much additional effort; we carry this out in Section 4. In addition, the proof of condition U for finite dimensional τ in the dense semisimple case is easier than a possible direct proof of the upper bound on the wavefront set by partial integration. Finally we are convinced that these uniformity conditions will also be useful in the future for proving upper bounds of the wavefront set for cases not covered in this article, for example for noncompact X and special infinite dimensional representations τ.

3. A Uniformity Condition and the Wavefront Set

Let G be a Lie group, let $H \subset G$ be a closed subgroup, and let (τ, V) be a unitary representation of H on a possibly infinite dimensional Hilbert space V. As in the introduction, we form the unitary representation $\operatorname{Ind}_H^G \tau$ of G. In Theorem 1.1 of [HHO], it is shown that

$$\mathrm{WF}(\operatorname{Ind}_H^G \tau) \supset \operatorname{Ind}_H^G \mathrm{WF}(\tau);\ \mathrm{SS}(\operatorname{Ind}_H^G(\tau)) \supset \operatorname{Ind}_H^G \mathrm{SS}(\tau).$$

In this section, we formulate a wavefront condition U on G, H, and τ, and we show that

$$\mathrm{WF}(\operatorname{Ind}_H^G \tau) \subset \operatorname{Ind}_H^G \mathrm{WF}(\tau)$$

when the wavefront condition U holds. In addition, we formulate an analogous singular spectrum condition U on G, H, and τ, and we show that

$$\mathrm{SS}(\operatorname{Ind}_H^G \tau) \subset \operatorname{Ind}_H^G \mathrm{SS}(\tau)$$

when the singular spectrum condition U holds. In order to formulate the wave front condition U and the singular spectrum condition U, we require additional notation.

Let $X = G/H$ be the corresponding homogeneous space on which G acts transitively from the left. If $x \in X$, we denote by $G_x \subset G$ the stabilizer subgroup of the point x in G. Obviously we have $G_{eH} = H$ and for all x, G_x is conjugate to H. Let $\mathcal{V} = G \times_H V$ denote the G equivariant bundle on $X = G/H$ associated to (τ, V), and let $\mathcal{D}^{1/2} \to X$ denote the bundle of complex half densities on X (See Appendix A). The group G acts in the standard way from the left on $\mathcal{V} \otimes \mathcal{D}^{1/2}$ by $g[x, v \otimes z] = [gx, v \otimes z]$. This left action leads to a unitary representation on $L^2(X, \mathcal{V} \otimes \mathcal{D}^{1/2})$ which we denote by $\operatorname{Ind}_H^G \tau$.

Let $\mathcal{V}_x$ (resp. $\mathcal{D}_x^{1/2}$) denote the fiber of $\mathcal{V}$ (resp. $\mathcal{D}^{1/2}$) over $x \in X$. Then the left G action induces a representation of G_x on $\mathcal{V}_x$ (resp. $\mathcal{D}_x^{1/2}$) which we denote by τ_x (resp. σ_x). Note that when $x = eH$, τ_{eH} coincides with τ. We analogously denote σ_{eH} by σ. Observe that we have the formula

$$\sigma_{eH}(h) = |\det_{T_{eH}X}(dh_{|eH})|^{-1/2} \tag{3.1}$$

where $\det_{T_{eH}X}(dh_{|eH})$ is the determinant of the differential

$$dh_{|eH} : T_{eH}X \to T_{eH}X$$

(see (A.8)).

Let $\mathfrak{g}, \mathfrak{h}$, and $\mathfrak{g}_x$ denote the Lie algebras of G, H, and G_x respectively. Note that on the Lie algebra level there is a canonical embedding $\mathfrak{g}_x \hookrightarrow \mathfrak{g}$ as a subalgebra and on the dual side there is a canonical projection $q_x : i\mathfrak{g}^* \to i\mathfrak{g}_x^*$ which is defined by the restriction of a form in $i\mathfrak{g}^*$ to elements in $\mathfrak{g}_x \subset \mathfrak{g}$. Throughout the article, we fix an arbitrary scalar product on $\mathfrak{g}$ which allows us to identify Lie algebras with their adjoints and defines a unique Lebesgue measure on $\mathfrak{g}$ and all of its linear subspaces.

For $x = eH$ we will drop the subscript in q_x and write $q := q_{eH} : i\mathfrak{g}^* \to i\mathfrak{h}^*$. If $S \subset i\mathfrak{h}^*$ is a subset, recall from the introduction the notation

$$\operatorname{Ind}_H^G S = \overline{\mathrm{Ad}^*(G) \cdot q^{-1}(S)}.$$

In the sequel we will be particularly interested in $\operatorname{Ind}_H^G \mathrm{WF}(\tau)$. We first want to relate this quantity to the representations τ_x on the fibers over x. For this purpose,

we recall how the representations τ_x and their wavefront sets are related. Let $x \in X$ and fix g_x such that $x = g_x H$. Then conjugation by g_x defines an isomorphism

$$\mathrm{C}_{g_x} : H \to G_x,\ h \mapsto g_x h g_x^{-1}.$$

Additionally, the left action by g_x defines a Hilbert space isomorphism

$$g_x : V = \mathcal{V}_{eH} \to \mathcal{V}_x$$

and we obtain for $h \in H$

$$\tau(h) = g_x^{-1} \tau_x(\mathrm{C}_{g_x} h) g_x. \tag{3.2}$$

Thus, τ and $\tau_x \circ \mathrm{C}_{g_x}$ are equivalent representations. In the same way we obtain

$$\sigma(h) = g_x^{-1} \sigma_x(\mathrm{C}_{g_x} h) g_x. \tag{3.3}$$

For the wavefront sets one consequently obtains from [Hor83, Theorem 8.2.4]

$$\mathrm{WF}(\tau) = \mathrm{WF}(\tau_x \circ \mathrm{C}_{g_x}) = (\mathrm{Ad}(g_x))^*(\mathrm{WF}(\tau_x)). \tag{3.4}$$

Here $\mathrm{Ad}(g_x) : \mathfrak{h} \to \mathfrak{g}_x$ appears as the differential of C_{g_x} in the identity element $e \in H$ and by pullback it defines canonically an isomorphism $(\mathrm{Ad}(g_x))^* : i\mathfrak{g}_x^* \to i\mathfrak{h}^*$. The analogous statement for the singular spectrum

$$\mathrm{SS}(\tau) = \mathrm{SS}(\tau_x \circ \mathrm{C}_{g_x}) = (\mathrm{Ad}(g_x))^*(\mathrm{SS}(\tau_x)) \tag{3.5}$$

can be obtained from [Hor83, Theorem 8.5.1] (Hörmander uses the term *analytic wave front set* instead of singular spectrum and writes WF_{A} instead of SS in his book).

Pullback and coadjoint action are always compatible with the natural projections as recorded in the following Lemma.

Lemma 3.1. *If* $\mathrm{Ad}^*(g_x) : i\mathfrak{g}^* \to i\mathfrak{g}^*$ *is the coadjoint representation and* $(\mathrm{Ad}(g_x))^* : i\mathfrak{g}_x^* \to i\mathfrak{h}^*$ *the pullback map, then*

$$q \circ \mathrm{Ad}^*(g_x^{-1}) = (\mathrm{Ad}(g_x))^* \circ q_x.$$

Proof. If $\xi \in i\mathfrak{g}^*$ and $H \in \mathfrak{h}$, then we calculate

$$\begin{aligned} [q \circ \mathrm{Ad}^*(g_x^{-1})(\xi)](H) &= [\mathrm{Ad}^*(g_x^{-1})(\xi)](H) \\ &= \xi(\mathrm{Ad}(g_x)H) \\ &= [q_x(\xi)](\mathrm{Ad}(g_x)H) \\ &= [(\mathrm{Ad}(g_x))^* \circ q_x](H). \end{aligned}$$

□

From this Lemma, (3.4), and (3.5) we conclude

$$q_x^{-1}(\mathrm{WF}(\tau_x)) = \mathrm{Ad}^*(g_x) q^{-1}(\mathrm{WF}(\tau)).$$

and

$$q_x^{-1}(\mathrm{SS}(\tau_x)) = \mathrm{Ad}^*(g_x) q^{-1}(\mathrm{SS}(\tau)).$$

Finally we can express

$$\mathcal{W} := \mathrm{Ind}_H^G \mathrm{WF}(\tau) = \overline{\bigcup_{g \in G} \mathrm{Ad}^*(g) q^{-1}(\mathrm{WF}(\tau))} = \overline{\bigcup_{x \in X} q_x^{-1}(\mathrm{WF}(\tau_x))}$$

and

$$\mathcal{S} := \mathrm{Ind}_H^G \mathrm{SS}(\tau) = \overline{\bigcup_{g \in G} \mathrm{Ad}^*(g) q^{-1}(\mathrm{SS}(\tau))} = \overline{\bigcup_{x \in X} q_x^{-1}(\mathrm{SS}(\tau_x))}.$$

As a consequence for any $\eta \notin \mathcal{W}$ and for any $x \in X$ we have $q_x(\eta) \notin \mathrm{WF}(\tau_x)$ and by the definition of the wavefront set (see for instance page 254 of [Hor83]), there is a neighborhood $V_x \subset i\mathfrak{g}_x$ of $q_x(\eta)$ and a function $\varphi_x \in C_c^\infty(\mathfrak{g}_x)$ supported in a neighborhood of $0 \in \mathfrak{g}_x$ such that for any N there is $C_{N,x} > 0$ such that

$$\left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \varphi_x(Y) e^{\langle t\xi, Y\rangle} dY \right| \leq C_{N,x} \|v_1\|_{\mathcal{V}_x} \|v_2\|_{\mathcal{V}_x} t^{-N} \tag{3.6}$$

for $t > 0,\ v_1, v_2 \in \mathcal{V}_x$. Here dY is the Lebesgue measure on $\mathfrak{g}_x$ which is fixed by the choice of the metric on $\mathfrak{g}$. The estimate is uniform in $\xi \in V_x$; however, we have a-priori no information about the uniformity of these estimates in x.

As the induced representations are modeled on $\mathcal{V} \otimes \mathcal{D}^{1/2}$, we will not only have to consider the representations τ_x but $\tau_x \otimes \sigma_x$ on $\mathcal{V}_x \otimes \mathcal{D}_x^{1/2}$ and will be led to the study of

$$\left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \left((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\right) \varphi_x(Y) e^{\langle t\xi, Y\rangle} dY \right|_{\mathcal{D}_x}$$

which now takes values in the fiber $(\mathcal{D}^1_{\geq 0})_x$. From (3.1) and (3.3) we conclude that for every x, the factor $\left((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\right)$ is a smooth function on Y with values in $\mathcal{D}^1_x$ and consequently one immediately gets fast decay analogous to (3.6) with an x dependent constant $C_{x,N}$. However the presence of the additional factor might be an additional source of x dependence of these constants.

The following condition imposes a uniformity in $x \in X$ on the decay of these Fourier transforms of the matrix coefficients and we will see in Sections 4 and 5 that this condition holds in many important examples.

Definition 3.2 (Wavefront Condition U)**.** With the notation from above we say that *wavefront condition U* is satisfied if for all $\eta \notin \mathrm{Ind}_H^G \mathrm{WF}(\tau)$ there is a neighborhood $\Omega \subset i\mathfrak{g}^* \setminus \mathrm{Ind}_H^G \mathrm{WF}(\tau)$ of η and a neighborhood $U_{\mathrm{CU,WF}} \subset \mathfrak{g}$ around $0 \in \mathfrak{g}$ such that for all functions $\varphi \in C_c^\infty(U_{\mathrm{CU,WF}})$ and for all $N \geq 0$ there is a constant $C_{N,\varphi}$ such that

$$\begin{aligned} &\left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot \left((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\right) \cdot \varphi(Y) e^{\langle t\xi, Y\rangle} dY \right|_{\mathcal{D}^1_x} \\ &\qquad \leq C_{N,\varphi} (\|v_1\|_{\mathcal{V}_x} |z_1|_{\mathcal{D}_x^{1/2}}) \otimes (\|v_2\|_{\mathcal{V}_x} |z_2|_{\mathcal{D}_x^{1/2}}) t^{-N} \end{aligned} \tag{3.7}$$

for $t > 0$ and uniformly for all $\xi \in \Omega,\ x \in X,\ v_1, v_2 \in \mathcal{V}_x$, and $z_1, z_2 \in \mathcal{D}_x^{1/2}$. Additionally we demand that there is a constant C such that for any $x \in X$, any $Y \in \mathfrak{g}_x \cap U_{\mathrm{CU,WF}}$ and any $z \in \mathcal{D}_x^{1/2}$, we have

$$|\sigma_x(e^Y)z|_{\mathcal{D}_x^{1/2}} \leq C|z|_{\mathcal{D}_x^{1/2}}. \tag{3.8}$$

Remark 3.3. Note that (3.7) is formulated as an inequality in the fiber $(\mathcal{D}^1_{\geq 0})_x$. Using a local point of reference $0 \neq z \in (\mathcal{D}^1_{\geq 0})_x$ these equations can simply be reduced to an inequality of non-negative real numbers. The advantage of the formulation (3.7) is that the left side as well as the right side can be considered as sections in $\mathcal{D}^1_{\geq 0}$ and we can consider (3.7) as an inequality of sections which will allow us to bound the integrals over X using (A.7).

We require an analogous condition for the singular spectrum. For this definition, we need additional notation. Choose a basis $\{X_1,\dots,X_n\}$ of $\mathfrak{g}$, and for every multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$, define the differential operator

$$D^\alpha=\partial_{X_1}^{\alpha_1}\partial_{X_2}^{\alpha_2}\cdots\partial_{X_n}^{\alpha_n}$$

on $\mathfrak{g}$. If $0\in U_1\subset U_2\subset\mathfrak{g}$ are precompact, open sets with $\overline{U_1}\subset U_2$, then (see pages 25-26, 282 of [Hor83]), we may find a sequence of functions $\{\varphi_{N,U_1,U_2}\}$ indexed by $N\in\mathbb{N}$ satisfying the following properties:

(1) $\varphi_{N,U_1,U_2}\in C_c^\infty(U_2)$
(2) $\varphi_{N,U_1,U_2}(x)=1$ if $x\in U_1$
(3) There exist constants $C_\alpha>0$ for every multi-index $\alpha=(\alpha_1,\dots,\alpha_n)$ such that

$$|D^{\alpha+\beta}\varphi_{N,U_1,U_2}(x)|\le C_\alpha^{|\beta|+1}(N+1)^{|\beta|}$$

for every $x\in U_1$ and every multi-index $\beta=(\beta_1,\dots,\beta_n)$. Here $|\beta|=\beta_1+\cdots+\beta_n$.

From now on, we fix a basis $\{X_1,\dots,X_n\}$ of $\mathfrak{g}$ and for every pair of precompact, open sets $0\in U_1\subset U_2\subset\mathfrak{g}$ with $\overline{U_1}\subset U_2$, we fix a sequence of functions $\{\varphi_{N,U_1,U_2}\}$ satisfying the above properties.

Definition 3.4 (Singular Spectrum Condition U)**.** With the notation from above we say that *singular spectrum condition U* is satisfied if for all $\eta\notin\operatorname{Ind}_H^G\operatorname{SS}(\tau)$ there is a neighborhood $\Omega\subset i\mathfrak{g}^*\setminus\operatorname{Ind}_H^G\operatorname{SS}(\tau)$ of η and a neighborhood $U_{\mathrm{CU,SS}}\subset\mathfrak{g}$ around $0\in\mathfrak{g}$ such that for every pair of neighborhoods $0\in U_1\Subset U_2\subset\mathfrak{g}$ with $U_2\subset U_{\mathrm{CU,SS}}$ and all $N\in\mathbb{N}$, there is a constant C_{U_1,U_2} such that

$$\left|\int_{\mathfrak{g}_x}\langle\tau_x(e^Y)v_1,v_2\rangle_{\mathcal{V}_x}\cdot\left((\sigma_x(e^Y)z_1)\otimes\overline{z_2}\right)\cdot\varphi_{N,U_1,U_2}(Y)e^{\langle t\xi,Y\rangle}dY\right|_{\mathcal{D}_x^1}\le C_{U_1,U_2}^{N+1}(N+1)^N(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}\tag{3.9}$$

for $t>0$ and uniformly for all $\xi\in\Omega$, $x\in X$, $v_1,v_2\in\mathcal{V}_x$, and $z_1,z_2\in\mathcal{D}_x^{1/2}$. Additionally we demand that there is a constant C such that for any $x\in X$, any $Y\in\mathfrak{g}_x\cap U_{\mathrm{CU,SS}}$ and any $z\in\mathcal{D}_x^{1/2}$, we have

$$|\sigma_x(e^Y)z|_{\mathcal{D}_x^{1/2}}\le C|z|_{\mathcal{D}_x^{1/2}}.\tag{3.10}$$

We then can show

Theorem 3.5. *Let G be a Lie group, let $H\subset G$ be a closed subgroup, and let τ be a unitary H representation. If wavefront condition U is satisfied, then we have*

$$\operatorname{WF}(\operatorname{Ind}_H^G\tau)\subset\operatorname{Ind}_H^G\operatorname{WF}(\tau).$$

In addition, we can show

Theorem 3.6. *Let G be a Lie group, let $H\subset G$ be a closed subgroup, and let τ be a unitary H representation. If singular spectrum condition U is satisfied, then we have*

$$\operatorname{SS}(\operatorname{Ind}_H^G\tau)\subset\operatorname{Ind}_H^G\operatorname{SS}(\tau).$$

We will spend the remainder of this section proving Theorem 3.5 and Theorem 3.6.

Let $U_{\mathrm{inj}} \subset \mathfrak{g}$ be an open neigbourhood of zero such that the exponential map,

$$\exp\colon U_{\mathrm{inj}} \subset \mathfrak{g} \to \exp(U_{\mathrm{inj}}) \subset G,$$

is a diffeomorphism and denote its inverse map by

$$\log : \exp(U_{\mathrm{inj}}) \subset G \to U_{\mathrm{inj}} \subset \mathfrak{g}.$$

For $f_1, f_2 \in L^2(X, \mathcal{V} \otimes \mathcal{D}^{1/2})$ and $\varphi \in C_c^\infty(G)$ with $\operatorname{supp}\varphi \subset \exp(U_{\mathrm{inj}})$, let us introduce the integral

$$I(f_1, f_2, \varphi, t\eta) = \int_G \left(\int_X \langle l_g f_1(g^{-1} \cdot x), f_2(x) \rangle_{\mathcal{V}_x \otimes \mathcal{D}_x^{1/2}} \right) \varphi(g) e^{\langle t\eta, \log(g) \rangle} dg.$$

Here dg is a non-zero, left invariant Haar measure. Note that

$$\langle l_g f_1(g^{-1} \cdot x), f_2(x) \rangle_{\mathcal{V}_x \otimes \mathcal{D}_x^{1/2}}$$

takes values in $\mathcal{D}^1$ and can thus be integrated over X.

If $\eta_0 \notin \mathcal{W} = \operatorname{Ind}_H^G \operatorname{WF}(\tau)$, then in order to prove Theorem 3.5 we have to show that $\eta_0 \notin \operatorname{WF}(\operatorname{Ind}_H^G \tau)$. Thus we have to show that for the chosen η_0, there is a cutoff function $\varphi \in C_c^\infty(G)$ with $\varphi(e) \neq 0$ and a neighborhood V_0 of η_0 such that the integral has fast decay in t uniformly in $\eta \in V_0$:

$$|I(f_1, f_2, \varphi, t\eta)| \leq C_{N, f_1, f_2, \varphi} |t|^{-N}. \tag{3.11}$$

On the other hand, in order to prove Theorem 3.6, we have to show that for a chosen $\eta_0 \notin \mathcal{S}$ there is a neighborhood V_0 of η_0 and two precompact, open neighborhoods $0 \in U_1 \Subset U_2 \subset U_{\mathrm{inj}} \subset \mathfrak{g}$ such that

$$|I(f_1, f_2, \log^* \varphi_{N, U_1, U_2}, t\eta)| \leq C_{f_1, f_2, \varphi}^{N+1} (N+1)^N |t|^{-N} \tag{3.12}$$

for every natural number $N \in \mathbb{N}$.

Recall that we fixed an inner product on $\mathfrak{g}$ and can identify $\mathfrak{g} \cong i\mathfrak{g}^*$ by this inner product. With Ω from Definition 3.2 (or Definition 3.4) there is $\varepsilon > 0$ such that $B_{2\varepsilon}(\eta_0)$, the ball of radius 2ε around η_0, is contained in Ω. We set

$$V_0 := B_\varepsilon(\eta_0).$$

In order to formulate the precise conditions on the uniform cutoff function φ which we will need in the proof of Theorem 3.5 and the conditions on the sets $0 \in U_1 \subset U_2 \subset \mathfrak{g}$ which we will need in the proof of Theorem 3.6, let us first formulate the following lemma. This lemma is a uniform non-stationary phase approximation and we will see below that together with wave front condition U and singular spectrum condition U, it will be one of the main ingredients in order to obtain the desired estimates (3.11, 3.12).

In what follows, whenever L is a Lie group with Lie algebra $\mathfrak{l}$ and $\exp\colon \mathfrak{l} \to L$ is the exponential map, we will write $e^X = \exp(X)$ for $X \in \mathfrak{l}$.

Lemma 3.7. *Let $\varepsilon > 0$. There is a neighborhood $U_{\mathrm{nsp}} \subset \mathfrak{g}$ of $0 \in \mathfrak{g}$ such that for any $\tilde{\varphi} \in C_c^\infty(U_{\mathrm{nsp}})$ and for any N, there is a constant $C_{N,\tilde{\varphi}}$ such that*

$$\left| \int\limits_{\mathfrak{g}_x} e^{t(\langle \xi, Y \rangle - \langle \eta, \log(e^Y e^Z) \rangle)} \tilde{\varphi}(Y) dY \right| \leq C_{N,\varphi} \langle \xi \rangle^{-N} |t|^{-N} \tag{3.13}$$

uniformly for $x \in X$, for all $\xi \in i\mathfrak{g}_x^* \setminus B_{2\varepsilon}(q_x(\eta_0))$, $Z \in U_{\mathrm{nsp}} \cap \mathfrak{g}_x^\perp$, and $\eta \in V_0$. In addition, if $0 \in \widetilde{U_1} \Subset \widetilde{U_2} \subset U_{\mathrm{nsp}} \subset \mathfrak{g}$, are open neighborhoods of zero, then there exists a constant $C_{\widetilde{U_1},\widetilde{U_2}}$ such that for every $N \in \mathbb{N}$,

$$\left| \int_{\mathfrak{g}_x} e^{t(\langle \xi, Y\rangle - \langle \eta, \log(e^Y e^Z)\rangle)} \varphi_{N,\widetilde{U_1},\widetilde{U_2}}(Y) dY \right| \leq C_{\widetilde{U_1},\widetilde{U_2}}^{N+1} (N+1)^N \langle \xi \rangle^{-N} |t|^{-N} \tag{3.14}$$

uniformly for $x \in X$, for all $\xi \in i\mathfrak{g}_x^* \setminus B_{2\varepsilon}(q_x(\eta_0))$, $Z \in U_{\mathrm{nsp}} \cap \mathfrak{g}_x^\perp$, and $\eta \in V_0 = B_\varepsilon(\eta_0)$.

Proof. For any $(\dim \mathfrak{h})$-dimensinal linear subspace $V \subset \mathfrak{g}$ we define

$$\psi_\eta^V : \mathfrak{g} = V \oplus V^\perp \to i\mathbb{R},\ Y + Z \mapsto \langle \eta, \log(e^Y e^Z)\rangle.$$

If

$$D_Y : C^\infty(V, i\mathbb{R}) \to C^\infty(V, iV^*)$$

is the differential, then we clearly have

$$D_Y \psi_\eta^V(Y, 0) = q_V(\eta)$$

where $q_V : i\mathfrak{g}^* \to iV^*$ is the restriction map. As $D_Y\psi_\eta^V(Y,Z)$ depends analytically on Z, for each V there is an open neighborhood of zero $U_V \subset \mathfrak{g}$ such that

$$\left\| D_Y \psi_\eta^V(Y,Z) - q_V(\eta) \right\|_{iV^*} < \frac{\varepsilon}{2} \tag{3.15}$$

for all $Z \in U_V \cap V^\perp$. As the Grassmanian of all $\dim \mathfrak{h}$-dimensional subspaces V is compact we can set

$$U_{\mathrm{nsp}} := \bigcap_{V \subset \mathfrak{g}} U_V$$

which is an open neighborhood of zero.

We now perform a partial integration via the differential operator

$$L^x = t^{-1} \frac{\langle \xi - D_Y \psi_\eta^{\mathfrak{g}_x}(Y,Z), D_Y \rangle_{i\mathfrak{g}_x^*}}{\langle \xi - D_Y \psi_\eta^{\mathfrak{g}_x}(Y,Z), \xi - D_Y \psi_\eta^{\mathfrak{g}_x}(Y,Z) \rangle_{i\mathfrak{g}_x^*}}$$

which has the property

$$L^x e^{t(\langle \xi, Y\rangle - \psi_\eta^{\mathfrak{g}_x}(Y,Z)\rangle)} = e^{t(\langle \xi, Y\rangle - \psi_\eta^{\mathfrak{g}_x}(Y,Z)\rangle)}.$$

Using (3.15) and the fact that $|\xi - q_x(\eta)|_{i\mathfrak{g}_x^*} > \varepsilon$ for all $\eta \in V_0$ and $\xi \in i\mathfrak{g}_x^* \setminus B_{2\varepsilon}(q_x(\eta_0))$, we calculate

$$\begin{aligned} \left\| \xi - D_Y \psi_\eta^{\mathfrak{g}_x}(Y,Z) \right\|_{i\mathfrak{g}_x^*} &\geq \left\| \xi - q_x(\eta) \right\|_{i\mathfrak{g}_x^*} - \left\| q_x(\eta) - D_Y \psi_\eta^{\mathfrak{g}_x}(Y,Z) \right\|_{i\mathfrak{g}_x^*} \\ &\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned} \tag{3.16}$$

So L^x is well defined in the desired ξ and η range. Performing N-times partial integration via L^x on the integral on the left hand side of (3.13) yields

$$\left| \int_{\mathfrak{g}_x} e^{t(\langle \xi, Y\rangle - \langle \eta, \log(e^Y e^Z)\rangle)} \tilde{\varphi}(Y) dY \right| \leq C_{N,\tilde{\varphi}} \left\| \xi - D_Y \psi_\eta^{\mathfrak{g}_x}(Y,Z) \right\|_{i\mathfrak{g}_x^*}^{-N} |t|^{-N}. \tag{3.17}$$

Finally using the observation that there is a constant $c > 0$ such that

$$|\xi - q_x(\eta)|_{i\mathfrak{g}_x^*} \geq c\langle \xi \rangle$$

uniform in $x \in X, \xi \in \mathfrak{g}_x^* \setminus B_{2\varepsilon}(q_x(\eta_0)), \eta \in V_0$ as well as (3.16) we obtain (3.13). In order to obtain the stronger statement (3.14), we must keep track of the constants in the partial integration more carefully. Utilizing statement (3) from the remarks preceding Definition 3.4 and the fact that L^x is a first order differentiable operator, we obtain bounds

$$|(L^x)^N \varphi_{N,\widetilde{U_1},\widetilde{U_2}}(y)| \leq C_{\widetilde{U_1},\widetilde{U_2}}^{N+1}(N+1)^N \left\|\xi - D_Y \psi_\eta^{\mathfrak{g}_x}(Y,Z)\right\|_{i\mathfrak{g}_x^*}^{-N} |t|^{-N}$$

for some constant $C_{\widetilde{U_1},\widetilde{U_2}}$ and all $y \in U_2$. Plugging this into (3.17) after replacing $\tilde{\varphi}$ with $\varphi_{N,\widetilde{U_1},\widetilde{U_2}}$ yields (3.14). □

Now consider for any $(\dim \mathfrak{h})$-dimensional subvectorspace $V \subset \mathfrak{g}$ the map

$$\kappa_V : \mathfrak{g} = V \oplus V^\perp \to G,\ Y + Z \mapsto e^Y e^Z.$$

Note that this map depends continuously on V as a point in the Grassmanian and as this Grassmanian is compact, there is a neighborhood of zero $U'_{\text{inj}} \subset \mathfrak{g}$ such that $(\kappa_V)_{|U'_{\text{inj}}}$ is a diffeomorphism for all V. Now, in the proof of Theorem 3.5 let $\tilde{\varepsilon} > 0$ be such that

$$B_{\tilde{\varepsilon}}(0) \Subset U_{\text{nsp}} \cap U'_{\text{inj}} \cap U_{\text{CU,WF}}. \tag{3.18}$$

And in the proof of Theorem 3.6 let $\tilde{\varepsilon} > 0$ be such that

$$B_{\tilde{\varepsilon}}(0) \Subset U_{\text{nsp}} \cap U'_{\text{inj}} \cap U_{\text{CU,SS}}. \tag{3.19}$$

Again from the compactness of the Grassmanian we conclude that $\bigcap_V \kappa_V(B_{\tilde{\varepsilon}}(0)) \subset G$ is a nonempty open neighborhood of $e \in G$. With this observation we can find a cutoff function $\varphi \in C_c^\infty(G)$ such that $\varphi(e) \neq 0$ and

$$\operatorname{supp} \varphi \subset \left(\bigcap_V \kappa_V(B_{\tilde{\varepsilon}}(0)) \right) \subset G \tag{3.20}$$

to be used in the proof of Theorem 3.5. Analogously, we may choose $0 \in U_1 \Subset U_2 \subset \mathfrak{g}$ precompact, open subsets such that

$$\exp(U_2) \subset \left(\bigcap_V \kappa_V(B_{\tilde{\varepsilon}}(0)) \right) \subset G \tag{3.21}$$

to be used in the proof of Theorem 3.6. Interchanging the X and G integrals and replacing the integral over G for each $x \in X$ by an integral over $\mathfrak{g}_x \oplus \mathfrak{g}_x^\perp$ via the

diffeomorphism $\kappa_{\mathfrak{g}_x}$ we obtain

$$|I(f_1,f_2,\varphi,t\eta)| = \left| \int\limits_X \int\limits_G \langle l_g f_1(g^{-1}\cdot x), f_2(x)\rangle_{\mathcal{V}_x} \varphi(g) e^{\langle t\eta, \log(g)\rangle} dg \right|$$

$$\leq \int\limits_X \left| \int\limits_G \langle l_g f_1(g^{-1}\cdot x), f_2(x)\rangle_{\mathcal{V}_x} \varphi(g) e^{\langle t\eta, \log(g)\rangle} dg \right|_{\mathcal{D}_x^1}$$

$$\leq \int\limits_X \int\limits_{\mathfrak{g}_x^\perp} \left| \int\limits_{\mathfrak{g}_x} \langle l_{e^Y e^Z} f_1((e^Y e^Z)^{-1}\cdot x), f_2(x)\rangle_{\mathcal{V}_x} \varphi(e^Y e^Z) e^{\langle t\eta, \log(e^Y e^Z)\rangle} j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(Y+Z) dY \right|_{\mathcal{D}_x^1} dZ$$

$$= \int\limits_X \int\limits_{\mathfrak{g}_x^\perp} \left| \int\limits_{\mathfrak{g}_x} \langle \tau_x\otimes\sigma_x(e^Y) l_{e^Z} f_1((e^Z)^{-1}\cdot x), f_2(x)\rangle_{\mathcal{V}_x} \varphi(e^Y e^Z) e^{\langle t\eta, \log(e^Y e^Z)\rangle} j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(Y+Z) dY \right|_{\mathcal{D}_x^1} dZ$$

where $j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}$ is the Jacobian of the diffeomorphism $\kappa_{\mathfrak{g}_x}$ and dY, dZ are Lebesgue measures on $\mathfrak{g}_x$ and $\mathfrak{g}_x^\perp$ which are fixed by the choice of the scalar product on $\mathfrak{g}$.

The analogous statement for the singular spectrum case is

$$|I(f_1,f_2,\log^*\varphi_{N,U_1,U_2},t\eta)|$$
$$\leq \int\limits_X \int\limits_{\mathfrak{g}_x^\perp} \left| \int\limits_{\mathfrak{g}_x} \langle \tau_x\otimes\sigma_x(e^Y) l_{e^Z} f_1((e^Z)^{-1}\cdot x), f_2(x)\rangle_{\mathcal{V}_x} \log^*\varphi_{N,U_1,U_2}(e^Y e^Z) e^{\langle t\eta, \log(e^Y e^Z)\rangle} j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(Y+Z) dY \right|_{\mathcal{D}_x^1} dZ.$$

Next for fixed $x\in X$ and $Z\in\mathfrak{g}_x^\perp$, we write $l_{e^Z} f_1(e^{-Z}\cdot x) = v_1\otimes z_1$ and $f_2(x) = v_2\otimes z_2$ with $v_i\in\mathcal{V}_x$ and $z_i\in\mathcal{D}_x^{1/2}$. We can then write the inner integral over $\mathfrak{g}_x$ as

$$\left\langle e^{\langle t\eta, \log(e^Y e^Z)\rangle}, \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^Y)z_1)\otimes\overline{z}_2\cdot\varphi(e^Y e^Z) j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(Y+Z)\right\rangle_{L^2(\mathfrak{g}_x)}$$

which takes values in $\mathcal{D}_x^1$ (For the singular spectrum case, we simply replace φ by $\log^*\varphi_{N,U_1,U_2}$). For the proof of Theorem 3.5, we choose a second cutoff function $\tilde\varphi\in C_c^\infty(\mathfrak{g})$ which fulfills

$$\begin{aligned} \mathrm{supp}(\tilde\varphi) &\subset U_{\mathrm{nsp}}\cap U_{\mathrm{CU,WF}}\cap U_{\mathrm{inj}} \\ \tilde\varphi(Y)=1 &\quad \text{for} \quad Y\in B_{\tilde\varepsilon}(0) \end{aligned}$$

which is possible because of the choice of $\tilde\varepsilon$ according to (3.18). For the proof of Theorem 3.6, we choose two open sets

$$\begin{aligned} \widetilde{U_2} &:= U_{\mathrm{nsp}}\cap U_{\mathrm{CU,SS}}\cap U_{\mathrm{inj}} \\ \widetilde{U_1} &:= B_{\tilde\varepsilon}(0) \end{aligned}$$

which is possible because of the choice of $\tilde{\varepsilon}$ according to (3.19). In the proof of Theorem 3.6, we will utilize the second cutoff function $\varphi_{N,\widetilde{U_1},\widetilde{U_2}}$ instead of $\widetilde{\varphi}$. From (3.20), we deduce that for all $x \in X$ and all $Y \notin B_{\tilde{\varepsilon}}(0) \cap \mathfrak{g}_x$ we have $\varphi(e^Y e^Z) = 0$ for all $Z \in \mathfrak{g}_x^\perp$. Thus, we can insert this new cutoff function $\tilde{\varphi}$ into our scalar product and calculate

$$
\begin{aligned}
&\left| \left\langle e^{\langle t\eta, \log(e^Y e^Z)\rangle} \tilde{\varphi}(Y), \langle \tau_x(e^Y) v_1, v_2 \rangle_{\mathcal{V}_x} \cdot (\sigma_x(e^Y) z_1) \otimes \overline{z}_2 \cdot \varphi(e^Y e^Z) j_{\mathfrak{g}_x, \mathfrak{g}_x^\perp}(Y + Z) \right\rangle_{L^2(\mathfrak{g}_x)} \right|_{\mathcal{D}_x^1} \\
&= \left| \left\langle \mathcal{F}\left[e^{\langle t\eta, \log(e^\bullet e^Z)\rangle} \tilde{\varphi}(\bullet) \right](\xi), \right.\right. \\
&\qquad \left.\left. \mathcal{F}\left[\langle \tau_x(e^\bullet) v_1, v_2 \rangle_{\mathcal{V}_x} \cdot (\sigma_x(e^\bullet) z_1) \otimes \overline{z}_2 \cdot \varphi(e^\bullet e^Z) j_{\mathfrak{g}_x, \mathfrak{g}_x^\perp}(\bullet + Z) \right](\xi) \right\rangle_{L^2(i\mathfrak{g}_x^*)} \right|_{\mathcal{D}_x^1} \\
&= |t|^{\dim \mathfrak{h}} \left| \left\langle \mathcal{F}\left[e^{\langle t\eta, \log(e^\bullet e^Z)\rangle} \tilde{\varphi}(\bullet) \right](t\xi), \right.\right. \\
&\qquad \left.\left. \mathcal{F}\left[\langle \tau_x(e^\bullet) v_1, v_2 \rangle_{\mathcal{V}_x} \cdot (\sigma_x(e^\bullet) z_1) \otimes \overline{z}_2 \cdot \varphi(e^\bullet e^Z) j_{\mathfrak{g}_x, \mathfrak{g}_x^\perp}(\bullet + Z) \right](t\xi) \right\rangle_{L^2(i\mathfrak{g}_x^*)} \right|_{\mathcal{D}_x^1} \\
&\leq |t|^{\dim \mathfrak{h}} \left\langle \left| \mathcal{F}\left[e^{\langle t\eta, \log(e^\bullet e^Z)\rangle} \tilde{\varphi}(\bullet) \right](t\xi) \right|, \right. \\
&\qquad \left. \left| \mathcal{F}\left[\langle \tau_x(e^\bullet) v_1, v_2 \rangle_{\mathcal{V}_x} \cdot (\sigma_x(e^\bullet) z_1) \otimes \overline{z}_2 \cdot \varphi(e^\bullet e^Z) j_{\mathfrak{g}_x, \mathfrak{g}_x^\perp}(\bullet + Z) \right](t\xi) \right|_{\mathcal{D}_x^1} \right\rangle_{L^2(i\mathfrak{g}_x^*)}
\end{aligned}
$$

For the singular spectrum case, the same expression holds with φ replaced by $\log^* \varphi_{N,U_1,U_2}$ and $\widetilde{\varphi}$ replaced by $\varphi_{N,\widetilde{U_1},\widetilde{U_2}}$. We will now split the integral over $i\mathfrak{g}_x^*$ into two parts, the first one over $B_{2\varepsilon}(q_x(\eta_0)) \subset i\mathfrak{g}_x^*$ and the second one over $i\mathfrak{g}_x^* \setminus B_{2\varepsilon}(q_x(\eta_0))$. The idea is to get the fast decay in t for the first integral by the wave front condition U (or, in the singular spectrum case, by singular spectrum condition U), and for the second part by the uniform nonstationary phase estimate (Lemma 3.7). Note that a similar strategy is used in the proof of [Dui73, Proposition 1.3.2]

First part $\int_{B_{2\varepsilon}(q_x(\eta_0))}$:
On the one side we have the trivial bound

$$
\left| \mathcal{F}\left[e^{\langle t\eta, \log(e^\bullet e^Z)\rangle} \tilde{\varphi}(\bullet) \right](t\xi) \right| \leq \int_{\mathfrak{g}_x} |\tilde{\varphi}(Y)| dY =: C_1. \tag{3.22}
$$

In the singular spectrum case, we utilize condition (3) satisfied by $\varphi_{N,\widetilde{U_1},\widetilde{U_2}}$ (see the remarks preceding Definition 3.4) with $\alpha = \emptyset$ and $\beta = \emptyset$ to obtain the trivial bound

(3.23)
$$
\left| \mathcal{F}\left[e^{\langle t\eta, \log(e^\bullet e^Z)\rangle} \varphi_{N,\widetilde{U_1},\widetilde{U_2}}(\bullet) \right](t\xi) \right| \leq \int_{\mathfrak{g}_x} |\varphi_{N,\widetilde{U_1},\widetilde{U_2}}(Y)| dY \leq \mathrm{vol}(\widetilde{U_2}) C_\emptyset =: C_1.
$$

On the other hand let us consider

$$\begin{aligned}&\left|\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\varphi(e^\bullet e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(\bullet+Z)\right](t\xi)\right|_{\mathcal{D}_x^1}=\\&\left|\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\tilde{\varphi}(\bullet)\varphi(e^\bullet e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(\bullet+Z)\right](t\xi)\right|_{\mathcal{D}_x^1}=\\&\left|\left(\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\tilde{\varphi}(\bullet)\right]*\mathcal{F}\left[\varphi(e^\bullet e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(\bullet+Z)\right]\right)(t\xi)\right|_{\mathcal{D}_x^1}\end{aligned}$$

Now wave front condition U implies

$$\begin{aligned}|\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\tilde{\varphi}(\bullet)\right](t\xi)|_{\mathcal{D}_x^1}&\leq\\C_N(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}&\end{aligned}\tag{3.24}$$

uniformly in $\xi\in B_{2\varepsilon}(q_x(\eta_0))\subset q_x(\Omega)$. The analogous singular spectrum statement is

$$\begin{aligned}\left|\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\varphi_{N,\widetilde{U_1},\widetilde{U_2}}(\bullet)\right](t\xi)\right|_{\mathcal{D}_x^1}&\leq\\C^{N+1}_{\widetilde{U_1},\widetilde{U_2}}(N+1)^N(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}&\end{aligned}\tag{3.25}$$

uniformly in $\xi\in B_{2\varepsilon}(q_x(\eta_0))\subset q_x(\Omega)$.

Next consider the family of functions $\rho_{x,Z}(Y):=\varphi(e^Ye^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(Y+Z)$ which can also be considered as a function of $Y\in\mathfrak{g}$. As the parameter $Z\in B_{\tilde{\varepsilon}}(0)$ can vary only in a bounded set due to the choice of φ according to (3.18) and (3.20) and as the dependence of $\rho_{x,Z}$ on x is only via the subvectorspace $\mathfrak{g}_x\subset\mathfrak{g}$ which can be considered as a point in the compact Grassmanian, $\rho_{x,Z}$ varies in a bounded set of $C^N(\mathfrak{g})$ for all $N\geq 0$. Accordingly, partial integration yields uniform fast decay of its Fourier transform in all directions, i.e.

$$\mathcal{F}\left[\varphi(e^\bullet e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(\bullet+Z)\right](\xi)\leq C_{N,\varphi}\langle\xi\rangle^{-N}\tag{3.26}$$

for all $\xi\in i\mathfrak{g}_x^*$ uniformly in $x\in X$ and $Z\in B_{\tilde{\varepsilon}}(0)\cap\mathfrak{g}_x^\perp$. The analogous statement for the singular spectrum case can be deduced from Lemma 6.2 of [HHO]. We have

$$\mathcal{F}\left[\varphi_{N,U_1,U_2}(e^\bullet e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(\bullet+Z)\right](\xi)\leq C^{N+1}_{U_1,U_2}(N+1)^N\langle\xi\rangle^{-N}\tag{3.27}$$

for all $\xi\in i\mathfrak{g}_x^*$ uniformly in $x\in X$ and $Z\in B_{\tilde{\varepsilon}}(0)\cap\mathfrak{g}_x^\perp$.

Now a straightforward calculation shows that the uniform decay property (3.24) is not destroyed by convolution with a function satisfying (3.26). So we obtain

$$\begin{aligned}|(\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\tilde{\varphi}(\bullet)\right]*\mathcal{F}\left[\rho_{x,Z}(\bullet)\right])(t\xi)|_{\mathcal{D}_x^1}&\\\leq C_{N,\varphi}(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}&\end{aligned}\tag{3.28}$$

uniformly in $\xi\in B_{2\varepsilon}(q_x(\eta_0))\subset q_x(\Omega)$.

For the singular spectrum case, we set $\rho_{x,Z}:=\varphi_{N,U_1,U_2}(e^Ye^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(Y+Z)$. Then, the uniform decay property (3.25) is not destroyed by convolution with a function satisfying (3.27), and we obtain

$$\begin{aligned}\left|\left(\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\varphi_{N,\widetilde{U_1},\widetilde{U_2}}(\bullet)\right]*\mathcal{F}\left[\rho_{x,Z}(\bullet)\right]\right)(t\xi)\right|_{\mathcal{D}_x^1}&\\\leq C^{N+1}_{U_1,U_2,\widetilde{U_1},\widetilde{U_2}}(N+1)^N(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}.&\end{aligned}\tag{3.29}$$

Now putting (3.22) and (3.28) together we obtain

$$\begin{aligned}(3.30)\qquad &\Bigg| \int\limits_{B_{2\varepsilon}(q_x(\eta_0))} \mathcal{F}\left[e^{\langle t\eta, \log(e^{\bullet}e^{Z})\rangle}\tilde{\varphi}(\bullet)\right](t\xi) \\ &\qquad \overline{\mathcal{F}\left[\langle \tau_x(e^{\bullet})v_1, v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^{\bullet})z_1)\otimes \overline{z}_2\cdot \rho_{x,Z}(\bullet)\right](t\xi)}d\xi\Bigg|_{\mathcal{D}_x^1} \\ &\leq C_{N,\varphi}(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}\end{aligned}$$

uniformly in $x \in X$ and $Z \in B_{\tilde{\varepsilon}}(0)$.

To get the singular spectrum analogue, we combine (3.23) and (3.29) to obtain

$$\begin{aligned}(3.31)\qquad &\Bigg| \int\limits_{B_{2\varepsilon}(q_x(\eta_0))} \mathcal{F}\left[e^{\langle t\eta, \log(e^{\bullet}e^{Z})\rangle}\varphi_{N,\widetilde{U_1},\widetilde{U_2}}(\bullet)\right](t\xi) \\ &\qquad \overline{\mathcal{F}\left[\langle \tau_x(e^{\bullet})v_1, v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^{\bullet})z_1)\otimes \overline{z}_2\cdot \rho_{x,Z}(\bullet)\right](t\xi)}d\xi\Bigg|_{\mathcal{D}_x^1} \\ &\leq C^{N+1}_{U_1,U_2,\widetilde{U_1},\widetilde{U_2}}(N+1)^N(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}\end{aligned}$$

uniformly in $x \in X$ and $Z \in B_{\tilde{\varepsilon}}(0) \cap \mathfrak{g}_x^{\perp}$.

Second part $\int_{i\mathfrak{g}_x^* \setminus B_{2\varepsilon}(q_x(\eta_0))}$**:**
Both in the wavefront and singular spectrum cases, for $\xi \in i\mathfrak{g}_x^* \setminus B_{2\varepsilon}(q_x(0))$, Lemma 3.7 provides us with fast decay in t and ξ of the first factor in the considered scalar product. In order to obtain fast decay of the scalar product we thus need a general bound for the second factor

$$\mathcal{F}\left[\langle \tau_x(e^{\bullet})v_1, v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^{\bullet})z_1)\otimes \overline{z}_2\cdot \varphi(e^{\bullet}e^{Z})j_{\mathfrak{g}_x,\mathfrak{g}_x^{\perp}}(\bullet+Z)\right](t\xi)$$

in the wavefront case and

$$\mathcal{F}\left[\langle \tau_x(e^{\bullet})v_1, v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^{\bullet})z_1)\otimes \overline{z}_2\cdot \varphi_{N,U_1,U_2}(e^{\bullet}e^{Z})j_{\mathfrak{g}_x,\mathfrak{g}_x^{\perp}}(\bullet+Z)\right](t\xi)$$

in the singular spectrum case. First note that (3.8) from wave front condition U and (3.10) from singular spectrum condition U together with the unitarity of τ_x imply a general bound on the norm

$$\|\tau_x(e^Y)v_1\|_{\mathcal{V}_x}\cdot|\sigma_x(e^Y)z_1|_{\mathcal{D}_x^{1/2}} \leq C_{\tau,\sigma}\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}}$$

uniformly in $x \in X$ and $Y \in B_{\tilde{\varepsilon}}(0) \cap \mathfrak{g}_x$. Again arguing with the compactness of the Grassmanian, we obtain

$$\int\limits_{\mathfrak{g}_x}\left|\varphi(e^Y e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^{\perp}}(Y+Z)\right|dY \leq C_2$$

uniformly in $x \in X$ and $Z \in B_{\tilde{\varepsilon}}(0) \cap \mathfrak{g}_x^{\perp}$ in the wavefront case and

$$\int\limits_{\mathfrak{g}_x}\left|\varphi_{N,U_1,U_2}(e^Y e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^{\perp}}(Y+Z)\right|dY \leq C_2$$

uniformly in $x \in X$ and $Z \in B_{\tilde{\varepsilon}}(0) \cap \mathfrak{g}_x^{\perp}$ in the singular spectrum case. We also utilized statement (3) of the remarks before Definition 3.4 in the singular spectrum

case. Consequently, in the wavefront case we obtain

$$\begin{aligned}
&\left|\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\varphi(e^\bullet e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(\bullet+Z)\right](t\xi)\right|\\
&\leq\int\limits_{\mathfrak{g}_x}\|\tau_x(e^Y)v_1\|_{\mathcal{V}_x}\|v_2\|_{\mathcal{V}_x}\cdot|\sigma_x(e^Y)z_1|_{\mathcal{D}_x^{1/2}}\otimes|z_2|_{\mathcal{D}_x^{1/2}}\left|\varphi(e^Ye^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(Y+Z)\right|dY\\
&\leq C_{\tau,\sigma}C_2\|v_1\|_{\mathcal{V}_x}\|v_2\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}}\otimes|z_2|_{\mathcal{D}_x^{1/2}}
\end{aligned}$$

uniformly in $t>0,\xi\in i\mathfrak{g}_x^*,x\in X$, and $T\in B_{\tilde{\varepsilon}}(0)\cap\mathfrak{g}_x^\perp$ and in the singular spectrum case, we obtain

$$\begin{aligned}
&\left|\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\varphi_{N,U_1,U_2}(e^\bullet e^Z)j_{\mathfrak{g}_x,\mathfrak{g}_x^\perp}(\bullet+Z)\right](t\xi)\right|\\
&\leq C_{\tau,\sigma}C_2\|v_1\|_{\mathcal{V}_x}\|v_2\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}}\otimes|z_2|_{\mathcal{D}_x^{1/2}}
\end{aligned}$$

uniformly in $t>0,\xi\in i\mathfrak{g}_x^*,x\in X$, and $z\in B_{\tilde{\varepsilon}}(0)$. Thus, in the wavefront case we get

$$\begin{aligned}
&\Bigg|\int\limits_{i\mathfrak{g}_x^*\setminus B_{2\varepsilon}(q_x(\eta_0))}\mathcal{F}\left[e^{\langle t\eta,\log(e^\bullet e^Z)\rangle}\tilde{\varphi}(\bullet)\right](t\xi)\\
&\qquad\overline{\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\rho_{x,Z}(Y)\right](t\xi)}d\xi\Bigg|_{\mathcal{D}_x^1}\\
&\leq C_{N,\varphi}(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}
\end{aligned}\tag{3.32}$$

and in the singular spectrum case, we get

$$\begin{aligned}
&\Bigg|\int\limits_{i\mathfrak{g}_x^*\setminus B_{2\varepsilon}(q_x(\eta_0))}\mathcal{F}\left[e^{\langle t\eta,\log(e^\bullet e^Z)\rangle}\varphi_{N,\widetilde{U_1},\widetilde{U_2}}(\bullet)\right](t\xi)\\
&\qquad\overline{\mathcal{F}\left[\langle\tau_x(e^\bullet)v_1,v_2\rangle_{\mathcal{V}_x}\cdot(\sigma_x(e^\bullet)z_1)\otimes\overline{z}_2\cdot\rho_{x,Z}(Y)\right](t\xi)}d\xi\Bigg|_{\mathcal{D}_x^1}\\
&\leq C_{\widetilde{U_1},\widetilde{U_2}}^{N+1}(N+1)^N(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}.
\end{aligned}\tag{3.33}$$

End of the proof of Theorem 3.5 and Theorem 3.6
To finish the proof of Theorem 3.5, we put (3.30) and (3.32) together to obtain

$$\begin{aligned}
|I(f_1,f_2,\varphi,t\eta)| &\leq \int\limits_X\int\limits_{g_x^\perp\cap B_{\tilde{\varepsilon}}(0)}C_{N,\varphi}(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}})\otimes(\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N}dZ\\
&= \int\limits_X\int\limits_{\mathfrak{g}_x^\perp\cap B_{\tilde{\varepsilon}}(0)}C_{N,\varphi}\|l_{e^Z}f_1(e^{-Z}x)\|_{\mathcal{V}_x\otimes D_x^{1/2}}\otimes\|f_2(x)\|_{\mathcal{V}_x\otimes\mathcal{D}_x^{1/2}}t^{-N}dZ
\end{aligned}$$

uniformly in $\eta\in V_0$. Note that the compact support of the Z integral follows from the choice of the cutoff function φ according to (3.20). As the regular representation on $L^2(X,\mathcal{V}\otimes\mathcal{D}^{1/2})$ is unitary, $f_1\in L^2(X,\mathcal{V}\otimes\mathcal{D}^{1/2})$ implies that $\|l_{e^Z}f_1(e^{-Z}x)\|_{\mathcal{V}_x}\in L^2(X,\mathcal{D}_{\geq 0}^{1/2})$ and so we finally obtain

$$|I(f_1,f_2,\varphi,t\eta)|\leq C_{N,f_1,f_2,\varphi}|t|^{-N}$$

uniformly in $\eta\in V_0$ which finishes the proof of Theorem 3.5.

To finish the proof of Theorem 3.6, we put (3.31) and (3.33) together and make the analogous argument to obtain

$$|I(f_1, f_2, \varphi_{N,U_1,U_2}, t\eta)| \le C^{N+1}_{U_1,U_2,f_1,f_2}(N+1)^N|t|^{-N}$$

uniformly in $\eta \in V_0$.

4. Proof of condition U for compact X

In this section, we verify the following Proposition.

Proposition 4.1. *Suppose G is a Lie group, $H \subset G$ is a closed subgroup, (τ, V) is a unitary representation of H, and $X = G/H$ is compact. Then wavefront condition U (Definition 3.2) and singular spectrum condition U (Definition 3.4) are both satisfied for the triple (G, H, τ).*

When combined with Theorem 3.5, Theorem 3.6, and Theorem 1.1 of [HHO], this will imply Theorem 1.3.

First, we show (3.8) and the identical statement (3.10). To do this, we show that for any precompact, open subset $U \subset \mathfrak{g}$, there exists a constant C such that for every $x \in X$, every $Y \in \mathfrak{g}_x \cap U$, and any $z \in \mathcal{D}_x^{1/2}$

$$|\sigma_x(e^Y)z|_{\mathcal{D}_x^{1/2}} \le C|z|_{\mathcal{D}_x^{1/2}}.$$

For fixed $x \in X$ and $Y \in \mathfrak{g}_x$, the quotient

$$\frac{|\sigma_x(e^Y)z|_{\mathcal{D}_x^{1/2}}}{|z|_{\mathcal{D}_x^{1/2}}}$$

is independent of $z \in \mathcal{D}_x^{1/2} \setminus \{0\}$. For each $x \in X$, this quotient defines a continuous function on $\mathfrak{g}_x$; in particular, it must attain a maximum value C_x on $\overline{\mathfrak{g}_x \cap U}$. Moreover, since $x \mapsto \mathfrak{g}_x$ is a continuous function of X into the Grassmannian and U is an open set, we deduce that $x \mapsto C_x$ is a continuous function on the compact space X and therefore attains a maximum value C. We deduce

$$|\sigma_x(e^Y)z|_{\mathcal{D}_x^{1/2}} \le C|z|_{\mathcal{D}_x^{1/2}}$$

for every $x \in X$, $Y \in \mathfrak{g}_x \cap U$, and $z \in \mathcal{D}_x^{1/2}$. This is (3.8) and (3.10).

Next, we must verify (3.7) for the wavefront case and (3.9) for the singular spectrum case. For the wavefront case, we must show that for a fixed $\eta_0 \notin \operatorname{Ind}_H^G \operatorname{WF} \tau$, there exists a neighborhood $\Omega \subset i\mathfrak{g}^* \setminus \operatorname{Ind}_H^G \operatorname{WF}(\tau)$ of η_0 and an open subset $0 \in U_{\mathrm{CU,WF}} \subset \mathfrak{g}$ such that for all $\varphi \in C_c^\infty(U_{\mathrm{CU,WF}})$ and all $N \in \mathbb{N}$, there exists a constant $C_{N,\varphi}$ such that

$$\left|\int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot \left((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\right) \cdot \varphi(Y) e^{\langle t\xi, Y\rangle} dY\right|_{\mathcal{D}_x^1} \le C_{N,\varphi}(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}}) \otimes (\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}})t^{-N} \tag{4.1}$$

for $t > 0$ and uniformly for all $\xi \in \Omega$, $x \in X$, $v_1, v_2 \in \mathcal{V}_x$, and $z_1, z_2 \in \mathcal{D}_x^{1/2}$.

For the singular spectrum case, we must show that for a fixed $\eta_0 \notin \operatorname{Ind}_H^G \operatorname{SS}(\tau)$, there is a neighborhood $\Omega \subset i\mathfrak{g}^* \setminus \operatorname{Ind}_H^G \operatorname{SS}(\tau)$ of η_0 and a neighborhood $0 \in U_{\mathrm{CU,SS}} \subset$

$\mathfrak{g}$ such that for every pair of neighborhoods $0 \in U_1 \Subset U_2 \subset \mathfrak{g}$ with $U_2 \subset U_{\mathrm{CU,SS}}$ and all $N \in \mathbb{N}$, there is a constant C_{U_1,U_2} such that

$$\left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot \big((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\big) \cdot \varphi_{N,U_1,U_2}(Y) e^{\langle t\xi, Y\rangle} dY \right|_{\mathcal{D}^1_x} \tag{4.2}$$
$$\leq C^{N+1}_{U_1,U_2}(N+1)^N (\|v_1\|_{\mathcal{V}_x} |z_1|_{\mathcal{D}^{1/2}_x}) \otimes (\|v_2\|_{\mathcal{V}_x} |z_2|_{\mathcal{D}^{1/2}_x}) t^{-N}$$

for $t > 0$ and uniformly for all $\xi \in \Omega$, $x \in X$, $v_1, v_2 \in \mathcal{V}_x$, and $z_1, z_2 \in \mathcal{D}^{1/2}_x$.

In order to study the integrals (4.1) and (4.2), we utilize a calculation in Appendix A. By (A.8), if $x \in X$, then the group G_x acts on the one dimensional complex vector space $\mathcal{D}^{1/2}_x$ by the scalar

$$\sigma_x(g_x)z = |\det_{T_xX}(dg_{x|x})|^{-1/2} z \tag{4.3}$$

for all $z \in \mathcal{D}^{1/2}_x$ where $\det_{T_xX}(dg_x|_x)$ denotes the determinant of the differential map on the fibre T_xX

$$dg_{x|x} \colon T_xX \to T_xX.$$

Injecting this expression into the left hand side of (4.1) or (4.2) yields

$$\begin{aligned}
&\left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot \big((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\big) \cdot \varphi(Y) e^{\langle t\xi, Y\rangle} dY \right|_{\mathcal{D}^1_x} \\
&= \left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot |\det_{T_xX}(de^Y{}_{|x})|^{-1/2} \cdot z_1 \otimes \overline{z_2} \cdot \varphi(Y) e^{\langle t\xi, Y\rangle} dY \right|_{\mathcal{D}^1_x} \\
&= \left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot |\det_{T_xX}(de^Y{}_{|x})|^{-1/2} \cdot \varphi(Y) e^{\langle t\xi, Y\rangle} dY \right| \cdot |z_1|_{\mathcal{D}^{1/2}_x} \otimes |z_2|_{\mathcal{D}^{1/2}_x}.
\end{aligned}$$

Next, for each $x \in X$, we choose $g_x \in G$ such that $g_x H g_x^{-1} = G_x$. Then we conjugate our integral from $\mathfrak{g}_x$, the Lie algebra of G_x, to $\mathfrak{h}$, the Lie algebra of H.

$$\left|\int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot |\det_{T_xX}(de^Y{}_{|x})|^{-1/2} \cdot \varphi(Y)e^{\langle t\xi, Y\rangle} dY\right| \cdot |z_1|_{\mathcal{D}_x^{1/2}} \otimes |z_2|_{\mathcal{D}_x^{1/2}}$$

$$= \left|\int_{\mathfrak{h}} \langle \tau_x(e^{\mathrm{Ad}(g_x)\tilde{Y}})v_1, v_2\rangle_{\mathcal{V}_x} \cdot |\det_{T_xX}(de^{\mathrm{Ad}(g_x)\tilde{Y}}{}_{|x})|^{-1/2} \cdot \right.$$

$$\left. \varphi(\mathrm{Ad}(g_x)\tilde{Y})e^{\langle t\xi, \mathrm{Ad}(g_x)\tilde{Y}\rangle} j_{\mathrm{Ad}(g_x):\mathfrak{h}\to\mathfrak{g}_x}(\tilde{Y})d\tilde{Y}\right| \cdot |z_1|_{\mathcal{D}_x^{1/2}} \otimes |z_2|_{\mathcal{D}_x^{1/2}}$$

$$= \left|\int_{\mathfrak{h}} \langle g_x^{-1}\tau_x(C_{g_x}e^{\tilde{Y}})g_x g_x^{-1}v_1, g_x^{-1}v_2\rangle_V \cdot |\det_{T_{eH}X}(de^{\tilde{Y}}{}_{|eH})|^{-1/2} \cdot \right.$$

$$\left. \varphi(\mathrm{Ad}(g_x)\tilde{Y})e^{\langle t\xi, \mathrm{Ad}(g_x)\tilde{Y}\rangle} j_{\mathrm{Ad}(g_x):\mathfrak{h}\to\mathfrak{g}_x}(\tilde{Y})d\tilde{Y}\right| \cdot |z_1|_{\mathcal{D}_x^{1/2}} \otimes |z_2|_{\mathcal{D}_x^{1/2}}$$

$$= \left|\int_{\mathfrak{h}} \langle \tau(e^{\tilde{Y}})g_x^{-1}v_1, g_x^{-1}v_2\rangle_V \cdot |\det_{T_{eH}X}(de^{\tilde{Y}}{}_{|eH})|^{-1/2} \cdot \right.$$

$$\left. \varphi(\mathrm{Ad}(g_x)\tilde{Y})e^{\langle t(\mathrm{Ad}(g_x))^*\xi, \tilde{Y}\rangle} j_{\mathrm{Ad}(g_x):\mathfrak{h}\to\mathfrak{g}_x}(\tilde{Y})d\tilde{Y}\right| \cdot |z_1|_{\mathcal{D}_x^{1/2}} \otimes |z_2|_{\mathcal{D}_x^{1/2}}$$

$$= (\star)$$

In reading the above calculations, recall that we have a fixed inner product on $\mathfrak{g}$ which restricts to $\mathfrak{g}_x$ and $\mathfrak{h}$, determining the translation invariant densities dY and $d\tilde{Y}$ on these vector spaces. In the above calculation, we have used the Jacobian of the conjugation map $\mathrm{Ad}(g_x)$, which is defined by the expression

$$\mathrm{Ad}(g_x)^* dY = j_{\mathrm{Ad}(g_x):\ \mathfrak{h}\to\mathfrak{g}_x}(\tilde{Y})d\tilde{Y}.$$

In addition, as before, C_{g_x} denotes conjugation by g_x.

The following Lemma will be useful.

Lemma 4.2. *If $X = G/H$ is a compact homogenous space, then for any $x \in X$ we can choose a g_x such that $x = g_x H$ and that*

$$\{g_x, x \in X\} \subset G$$

is a precompact set.

Proof. Define the g_x via a finite number of continuous sections of the principle H-fiber bundle $G \to X$. □

Let us assume from now on that the g_x are fixed according to Lemma 4.2.

Now we are ready to prove wavefront condition U and singular spectrum condition U. For the wavefront case, let $\eta_0 \notin \mathcal{W} = \mathrm{Ind}_H^G \mathrm{WF}(\tau)$. As $\mathcal{W}$ is closed we can fix $\Omega = B_\varepsilon(\eta_0) \in i\mathfrak{g}^* \setminus \mathcal{W}$. Next let us introduce

$$V_{0,\mathrm{WF}} := \bigcup_{x\in X} (\mathrm{Ad}(g_x))^* q_x(\Omega) = \bigcup_{x\in X} q(\mathrm{Ad}^*(g_x^{-1})\Omega) \subset i\mathfrak{h}^*.$$

The importance of the set $V_{0,\mathrm{WF}}$ arises from the observation that for $\xi \in q_x(\Omega)$ we have $(\mathrm{Ad}(g_x))^*\xi \in V_{0,\mathrm{WF}}$. From the choice of g_x according to Lemma 4.2 we

conclude that $V_{0,\mathrm{WF}}$ is a precompact set. Additionally we get from the $\mathrm{Ad}^*(G)$ invariance of $\mathcal{W}$ that

$$\overline{\bigcup_{x\in X} \mathrm{Ad}^*(g_x^{-1})\Omega} \cap \mathcal{W} = \emptyset$$

and consequently

$$V_{0,\mathrm{WF}} \subset i\mathfrak{h}^* \setminus \mathrm{WF}(\tau)$$

is a precompact subset. Using the fact that $V_{0,\mathrm{WF}}$ is disjoint with $\mathrm{WF}(\tau)$ and that it is precompact, we deduce that there is $\tilde{\varepsilon}_{\mathrm{WF}} > 0$ such that for all smooth functions $\tilde{\varphi}$ with $\mathrm{supp}(\tilde{\varphi}) \subset B_{\tilde{\varepsilon}_{\mathrm{WF}}}(0) \subset \mathfrak{h}$ and all $N > 0$ there is a constant $C_{N,\tilde{\varphi}}$ uniformly in $\tilde{\xi} \in V_{0,\mathrm{WF}}$ such that

$$\left| \int_{\mathfrak{h}} \langle \tau(e^{\tilde{Y}})v_1, v_2\rangle_V \tilde{\varphi}(\tilde{Y}) e^{\langle t\tilde{\xi},\tilde{Y}\rangle} d\tilde{Y} \right| \leq C_{N,\tilde{\varphi}} \|v_1\| \cdot \|v_2\| t^{-N} \tag{4.4}$$

For the singular spectrum analogue, we define $\mathcal{S} := \mathrm{Ind}_H^G \mathrm{SS}(\tau)$, and we define $V_{0,\mathrm{SS}}$ with WF replaced by SS everywhere. We analogously obtain that there is $\tilde{\varepsilon}_{\mathrm{SS}} > 0$ such that whenever $0 \in \widetilde{U_1} \Subset \widetilde{U_2} \subset B_{\tilde{\varepsilon}_{\mathrm{SS}}}(0) \subset \mathfrak{h}$ are open sets, there exists a constant $C_{\widetilde{U_1},\widetilde{U_2}}$ for which

$$\left| \int_{\mathfrak{h}} \langle \tau(e^{\tilde{Y}})v_1, v_2\rangle_V \varphi_{N,\widetilde{U_1},\widetilde{U_2}}(\tilde{Y}) e^{\langle t\tilde{\xi},\tilde{Y}\rangle} d\tilde{Y} \right| \leq C^{N+1}_{\widetilde{U_1},\widetilde{U_2}} (N+1)^N \|v_1\| \cdot \|v_2\| t^{-N} \tag{4.5}$$

uniformly in $\tilde{\xi} \in V_{0,\mathrm{SS}}$.

Now we will fix the open neighborhood of $0 \in U_{CU,\mathrm{WF}} \subset \mathfrak{g}$ according to

$$U_{CU,\mathrm{WF}} \subset \bigcap_{x\in X} \mathrm{Ad}(g_x) B_{\tilde{\varepsilon}_{\mathrm{WF}}/2}(0) \subset \mathfrak{g}$$

Once more this is possible due to Lemma 4.2. Finally we choose a particular $\tilde{\varphi} \in C_0^\infty(B_{\tilde{\varepsilon}_{\mathrm{WF}}}(0))$ such that $\tilde{\varphi}(Y) = 1$ for $Y \in B_{\tilde{\varepsilon}_{\mathrm{WF}}/2}(0) \subset \mathfrak{h}$. Note that the cutoff function φ from above satisfies $\mathrm{supp}\,\varphi \subset U_{CU,\mathrm{WF}}$ and by the choice of $U_{CU,\mathrm{WF}}$ we conclude $\mathrm{supp}(\varphi \circ \mathrm{Ad}(g_x)) \subset B_{\tilde{\varepsilon}_{\mathrm{WF}}/2}(0)$. Thus we can insert $\tilde{\varphi}$ in $(\star)$ and obtain

$$\begin{aligned}
(\star) = &\Bigg| \mathcal{F}\Big[\langle \tau(e^\bullet) g_x^{-1} v_1, g_x^{-1} v_2\rangle_V \cdot \tilde{\varphi}(\bullet) \cdot |\mathrm{det}_{T_{eH}X}(de^\bullet{}_{|x})|^{-1/2} \\
&\cdot \varphi \circ \mathrm{Ad}(g_x)(\bullet) \cdot j_{\mathrm{Ad}\, g_x : \mathfrak{h} \to \mathfrak{g}_x}(\bullet) \Big] (t(\mathrm{Ad}(g_x))^* \xi) \Bigg| \cdot |z_1|_{\mathcal{D}_x^{1/2}} \otimes |z_2|_{\mathcal{D}_x^{1/2}} \\
= &\left| \left(\mathcal{F}\Big[\langle \tau(e^\bullet) g_x^{-1} v_1, g_x^{-1} v_2\rangle_V \cdot \tilde{\varphi}(\bullet) \Big] * \mathcal{F}\Big[\rho_{x,\varphi}(\bullet) \Big] \right) (t(\mathrm{Ad}(g_x))^* \xi) \right| \\
&\cdot |z_1|_{\mathcal{D}_x^{1/2}} \otimes |z_2|_{\mathcal{D}_x^{1/2}}
\end{aligned}$$

where

$$\rho_{x,\varphi}(\tilde{Y}) = |\mathrm{det}_{T_{eH}X}(de^{\tilde{Y}}{}_{|x})|^{-1/2} \cdot \varphi(\mathrm{Ad}(g_x)\tilde{Y}) \cdot j_{g_x : \mathfrak{h} \to \mathfrak{g}_x}(\tilde{Y}).$$

For the singular spectrum case, we analogously define $U_{CU,\mathrm{SS}}$ utilizing $\tilde{\varepsilon}_{\mathrm{SS}}$ and we obtain the above statement with φ replaced by φ_{N,U_1,U_2} where $0 \in U_1 \Subset U_2 \subset U_{CU,\mathrm{SS}}$ and with $\tilde{\varphi}$ replaced by $\varphi_{N,\widetilde{U_1},\widetilde{U_2}}$ where $0 \in \widetilde{U_1} \Subset \widetilde{U_2} \subset B_{\tilde{\varepsilon}_{\mathrm{SS}}}(0)$.

Back in the wavefront case, we observe that from the compactness of X we obtain that

$$|\mathcal{F}[\rho_{x,\varphi}](\xi)| \le C_{N,\varphi}\langle\xi\rangle^{-N}$$

with $C_{N,\varphi}$ is independent of $x \in X$. Using this observation together with (4.4) we finally obtain

$$\begin{aligned}(\star) &\le C_{N,\varphi}\|g_x^{-1}v_1\|\|g_x^{-1}v_2\| \cdot |z_1| \otimes |z_2||t|^{-N}\\ &= C_{N,\varphi}\|v_1\|\|v_2\| \cdot |z_1| \otimes |z_2||t|^{-N}.\end{aligned}$$

We have thus shown (3.7) and verified wavefront condition U. For the singular spectrum case, we replace φ by φ_{N,U_1,U_2} in the definition of $\rho_{x,\varphi}$ (which now depends on N) and we utilize the stronger bounds

$$|\mathcal{F}[\rho_{x,\varphi}](\xi)| \le C_{U_1,U_2}^{N+1}(N+1)^N\langle\xi\rangle^{-N}$$

where C_{U_1,U_2} is a constant independent of N. These stronger bounds can be obtained from a boundary values of holomorphic functions argument which is similar to (though not identical to) the proof of Lemma 6.2 of [HHO]. Finally the bound (3.10) follows directly with the compactness of X. This finishes the proof of singular spectrum condition U.

5. Proof of condition U in the dense semisimple case

5.1. **The dense semisimple condition.** Suppose G is a real, linear algebraic group, and let $H \subset G$ be a closed subgroup. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) denote the Lie algebra of G (resp. H). Assume that there exists a real, linear algebraic group $H_1 \subset G$ with Lie algebra $\mathfrak{h}$ (Note that we do not assume H is algebraic).

Definition 5.1. We say $X \in \mathfrak{h}$ is semisimple if and only if for every homomorphism of algebraic groups

$$\rho\colon H_1 \to \mathrm{GL}(N,\mathbb{R}),$$

$d\rho(X) \in \mathfrak{gl}(N,\mathbb{R})$ is diagonalizable over the complex numbers.

By the general theory of linear, algebraic groups, to check whether X is semisimple, it is enough to find one *injective* map

$$\rho\colon H_1 \to \mathrm{GL}(N,\mathbb{R})$$

and check whether $d\rho(X)$ is semisimple (see for instance 4.4 Theorem on pages 83 and 84 of [Bor91]). Moreover, we note that $X \in \mathfrak{h} \subset \mathfrak{g}$ is semisimple with respect to H_1 iff it is semisimple with respect to G.

Definition 5.2. Let $\mathfrak{h}_s \subset \mathfrak{h}$ denote the cone of elements of $\mathfrak{h}$ that are semisimple with respect to H_1.

We wish to study the case where $\mathfrak{h}_s \subset \mathfrak{h}$ is dense and prove the following theorem.

Theorem 5.3. *Suppose G is a real, linear algebraic group, suppose $H \subset G$ is a closed subgroup, and suppose τ is a finite dimensional, unitary representation of H. Assume that there exists a real, linear algebraic group $H_1 \subset G$ such that $\mathfrak{h}$, the Lie algebra of H, is also the Lie algebra of $\mathfrak{h}_1$. In addition, assume $\mathfrak{h}_s \subset \mathfrak{h}$ is dense. Then wavefront condition U and singular spectrum condition U are both satisfied.*

Note that Theorem 5.3 together with Theorem 3.5 (resp. Thm 3.6 in the singular spectrum case) and [HHO, Theorem 1.1] imply Theorem 1.2.

If H is a real, reductive algebraic group, then $\mathfrak{h}$ contains an open, dense subset of semisimple elements. If $H = P$ is a parabolic subgroup of a real, reductive algebraic group, then $\mathfrak{h} = \mathfrak{p}$ has a dense subset of semisimple elements. In particular, this holds when $H = B$ and

$$B = \left\{ \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \;\Big|\; a \in \mathbb{R}^\times,\ x \in \mathbb{R} \right\}$$

denotes the motion group of the real line. If $H = N$ is an infinite unipotent group, for instance

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\Big|\; x \in \mathbb{R} \right\},$$

then $\mathfrak{h}_s = \mathfrak{n}_s = \{0\}$ and $\mathfrak{h} = \mathfrak{n}$ does not have a dense subset of semisimple elements.

5.2. **On the conjugacy of maximal toral subalgebras in real, linear algebraic groups.** In order to prove Theorem 5.3, we first need a technical fact from the structure theory of real, linear algebraic groups.

Let $H_\mathbb{C}$ be a connected, complex linear algebraic group defined over $\mathbb{R}$, and let $H = H_\mathbb{C}(\mathbb{R})$ be the real points of $H_\mathbb{C}$.

Definition 5.4. We say that a complex linear algebraic group $T_\mathbb{C}$ is *diagonalizable* if it is isomorphic to a closed subgroup of a product of copies of $\mathbb{C}^\times$. A *torus* $T_\mathbb{C} \subset H_\mathbb{C}$ in $H_\mathbb{C}$ is a connected, diagonalizable subgroup of $H_\mathbb{C}$. A *maximal torus* in $H_\mathbb{C}$ is a torus in $H_\mathbb{C}$ that is not properly contained in another torus in $H_\mathbb{C}$.

Proposition 5.5. *There are finitely many H conjugacy classes of maximal tori $T_\mathbb{C} \subset H_\mathbb{C}$ which are defined over $\mathbb{R}$.*

Proof. Let $(H_\mathbb{C})_u$ denote the unipotent radical of $H_\mathbb{C}$. It is the unique maximal closed, connected, normal, unipotent subgroup of $H_\mathbb{C}$ (see for instance 11.21 on page 157 of [Bor91]). Moreover, $(H_\mathbb{C})/(H_\mathbb{C})_u$ is a complex, reductive algebraic group. Now, we assumed that $H_\mathbb{C}$ was defined over $\mathbb{R}$ with real form H. It follows that $(H_\mathbb{C})_u$ is defined over $\mathbb{R}$ (see 14.4.5 Proposition on page 250 of [Spr98]); we will denote the corresponding set of real points by H_u. By Corollary 12.2.2 on page 212 of [Spr98], we see that H/H_u is a real form of $H_\mathbb{C}/(H_\mathbb{C})_u$. In particular, H/H_u is a real, reductive algebraic group.

Let

$$\rho_\mathbb{C} \colon H_\mathbb{C} \to H_\mathbb{C}/(H_\mathbb{C})_u$$

denote the natural surjective homomorphism. By 11.14 Proposition (1) on page 152 of [Bor91], $\rho_\mathbb{C}$ induces a map from maximal tori in $H_\mathbb{C}$ to maximal tori in $H_\mathbb{C}/(H_\mathbb{C})_u$. Note that this map is surjective. Indeed, suppose $B_\mathbb{C} \subset H_\mathbb{C}/(H_\mathbb{C})_u$ is a maximal torus in $H_\mathbb{C}/(H_\mathbb{C})_u$. Choose a maximal torus $T_\mathbb{C} \subset H_\mathbb{C}$, and note that $\rho_\mathbb{C}(T_\mathbb{C})$ is conjugate to $B_\mathbb{C}$ since all maximal tori in $H_\mathbb{C}/(H_\mathbb{C})_u$ are conjugate (see 11.3 Corollary (1) on page 148 of [Bor91]). In particular, there exists $\rho_\mathbb{C}(g) \in H_\mathbb{C}/(H_\mathbb{C})_u$ such that

$$\rho_\mathbb{C}(g)\rho_\mathbb{C}(T_\mathbb{C})\rho_\mathbb{C}(g)^{-1} = B_\mathbb{C}.$$

One sees

$$\rho_\mathbb{C}(gT_\mathbb{C}g^{-1}) = B_\mathbb{C}$$

and the induced map on maximal tori is surjective.

Now, we know that there are finitely many H conjugacy classes of maximal tori $B_{\mathbb{C}} \subset H_{\mathbb{C}}/(H_{\mathbb{C}})_u$ that are defined over $\mathbb{R}$ (this may be deduced from the definition of Cartan subalgebra on page 254 of [Kna05], the remarks on pages 457 and 458 of [Kna05] including Proposition 7.35, and the easy fact that a maximal torus is determined by its Lie algebra). Let $\{B_{\mathbb{C}}^{\alpha}\}_{\alpha\in\mathcal{A}}$ be a set of representatives for these conjugacy classes. For each α, choose a maximal torus $T_{\mathbb{C}}^{\alpha} \subset H_{\mathbb{C}}$ such that

$$\rho_{\mathbb{C}}(T_{\mathbb{C}}^{\alpha}) = B_{\mathbb{C}}^{\alpha}.$$

We claim that every maximal torus $T_{\mathbb{C}} \subset H_{\mathbb{C}}$ that is defined over $\mathbb{R}$ is H conjugate to a maximal torus of the form $T_{\mathbb{C}}^{\alpha}$. Indeed, $\rho_{\mathbb{C}}(T_{\mathbb{C}})$ is defined over $\mathbb{R}$; hence, there exists $g \in H$ such that

$$\rho_{\mathbb{C}}(g)\rho_{\mathbb{C}}(T_{\mathbb{C}})\rho_{\mathbb{C}}(g)^{-1} = B_{\mathbb{C}}^{\alpha}$$

for some α. In particular, $gT_{\mathbb{C}}g^{-1} \in \rho_{\mathbb{C}}^{-1}(B_{\mathbb{C}}^{\alpha})$.

Next, we observe

$$\rho_{\mathbb{C}}^{-1}(B_{\mathbb{C}}^{\alpha}) = T_{\mathbb{C}}^{\alpha} \cdot (H_{\mathbb{C}})_u$$

since $(H_{\mathbb{C}})_u \subset H_{\mathbb{C}}$ is a normal subgroup. One deduces that $\rho_{\mathbb{C}}^{-1}(B_{\mathbb{C}}^{\alpha})$ is a solvable subgroup of $H_{\mathbb{C}}$ that is defined over $\mathbb{R}$. By 19.2 Theorem on page 223 of [Bor91], all maximal tori in $\rho_{\mathbb{C}}^{-1}(B_{\mathbb{C}}^{\alpha})$ that are defined over $\mathbb{R}$ are conjugate via the set of real points of $\rho_{\mathbb{C}}^{-1}(B_{\mathbb{C}}^{\alpha})$ (in particular, they are conjugate via H). Since $T_{\mathbb{C}}^{\alpha}$ is a maximal torus in $\rho_{\mathbb{C}}^{-1}(B_{\mathbb{C}}^{\alpha})$ that is defined over $\mathbb{R}$, we deduce that $gT_{\mathbb{C}}g^{-1}$ is conjugate to $T_{\mathbb{C}}^{\alpha}$. Thus, $T_{\mathbb{C}}$ is conjugate to $T_{\mathbb{C}}^{\alpha}$. The Proposition follows. □

Next, we move to the Lie algebra. Let $\mathfrak{h}$ denote the Lie algebra of H, and let $\mathfrak{h}_{\mathbb{C}}$ denote the Lie algebra of $H_{\mathbb{C}}$

Definition 5.6. A *toral subalgebra* $\mathfrak{t}_{\mathbb{C}}$ in $\mathfrak{h}_{\mathbb{C}}$ is an abelian subalgebra consisting of semisimple elements. A *maximal toral subalgebra* of $\mathfrak{h}_{\mathbb{C}}$ is a toral subalgebra of $\mathfrak{h}_{\mathbb{C}}$ that is not properly contained in another toral subalgebra of $\mathfrak{h}_{\mathbb{C}}$.

Lemma 5.7. *Suppose $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{h}_{\mathbb{C}}$ is a maximal toral subalgebra. Then there exists a maximal torus $T_{\mathbb{C}} \subset G_{\mathbb{C}}$ such that $\mathfrak{t}_{\mathbb{C}}$ is the Lie algebra of $T_{\mathbb{C}}$. In particular, all maximal toral subalgebras of $\mathfrak{h}_{\mathbb{C}}$ are conjugate.*

In addition, if $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{h}_{\mathbb{C}}$ is a maximal toral subalgebra defined over $\mathbb{R}$, then the corresponding maximal torus $T_{\mathbb{C}} \subset H_{\mathbb{C}}$ is defined over $\mathbb{R}$. In particular, there are finitely many H conjugacy classes of maximal toral subalgebras of $\mathfrak{h}_{\mathbb{C}}$ that are defined over $\mathbb{R}$.

Proof. Let $B_{\mathbb{C}} = Z_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$ be the centralizer in $H_{\mathbb{C}}$ of $\mathfrak{t}_{\mathbb{C}}$, and note

$$\mathfrak{b}_{\mathbb{C}} := \mathrm{Lie}(B_{\mathbb{C}}) = Z_{\mathfrak{h}_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}}).$$

Let $\widetilde{T_{\mathbb{C}}} \subset B_{\mathbb{C}}$ be a maximal torus, and let $C_{\mathbb{C}} = Z_{B_{\mathbb{C}}}(\widetilde{T_{\mathbb{C}}})$ be the associated Cartan subgroup of $B_{\mathbb{C}}$. Then

$$\mathfrak{c}_{\mathbb{C}} := \mathrm{Lie}(Z_{B_{\mathbb{C}}}(\widetilde{T_{\mathbb{C}}})) = Z_{\mathfrak{b}_{\mathbb{C}}}(\mathrm{Lie}(\widetilde{T_{\mathbb{C}}}))$$

Hence, $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{c}_{\mathbb{C}}$ since $\mathfrak{t}_{\mathbb{C}}$ is in the center of $\mathfrak{b}_{\mathbb{C}}$, and therefore it must centralize $\mathrm{Lie}(\widetilde{T_{\mathbb{C}}}) \subset \mathfrak{b}_{\mathbb{C}}$. But, $C_{\mathbb{C}}$ is a nilpotent Lie group; hence, by part (3) of 10.6 Theorem on page 138 of [Bor91], we have

$$C_{\mathbb{C}} \simeq (C_{\mathbb{C}})_s \times (C_{\mathbb{C}})_u$$

and

$$\mathfrak{c}_{\mathbb{C}} \simeq (\mathfrak{c}_{\mathbb{C}})_s \oplus (\mathfrak{c}_{\mathbb{C}})_n.$$

In particular, we see that $\widetilde{T_{\mathbb{C}}} = (C_{\mathbb{C}})_s$ is the unique maximal toral subgroup of $C_{\mathbb{C}}$ and its Lie algebra $\widetilde{\mathfrak{t}_{\mathbb{C}}} = (\mathfrak{c}_{\mathbb{C}})_s$ is the unique maximal toral subalgebra of $\mathfrak{c}_{\mathbb{C}}$. Since $\mathfrak{t}_{\mathbb{C}}$ is a toral subalgebra in $\mathfrak{c}_{\mathbb{C}}$, it must be a subalgebra of $\widetilde{\mathfrak{t}_{\mathbb{C}}}$. But, $\mathfrak{t}_{\mathbb{C}}$ is maximal in $\mathfrak{g}_{\mathbb{C}}$; hence, it must be maximal in the smaller algebra $\mathfrak{c}_{\mathbb{C}}$. Therefore,

$$\mathfrak{t}_{\mathbb{C}} = \widetilde{\mathfrak{t}_{\mathbb{C}}}$$

and $\mathfrak{t}_{\mathbb{C}}$ is the Lie algebra of the maximal torus $\widetilde{T_{\mathbb{C}}} \subset B_{\mathbb{C}}$. However, any torus containing $\widetilde{T_{\mathbb{C}}}$ in $H_{\mathbb{C}}$ would have to centralize $\widetilde{T_{\mathbb{C}}}$; hence, it would have to lie in $B_{\mathbb{C}}$. Therefore $\mathfrak{t}_{\mathbb{C}}$ is the Lie algebra of a maximal torus in $H_{\mathbb{C}}$.

The second statement follows from the conjugacy of maximal tori and the fact that if two groups are conjugate, then their Lie algebras must also be conjugate.

For the third statement, we assume $\mathfrak{t}_{\mathbb{C}}$ is defined over $\mathbb{R}$. Then its centralizer $B_{\mathbb{C}}$ must be defined over $\mathbb{R}$. Now, a complex, linear algebraic group defined over $\mathbb{R}$ always has a maximal torus defined over $\mathbb{R}$ (see 18.2 Theorem (i) on page 182 of [Bor91]). Then we may choose $\widetilde{T_{\mathbb{C}}}$ in the above argument to be defined over $\mathbb{R}$, and we have a maximal torus with Lie algebra $\mathfrak{t}_{\mathbb{C}}$ that is defined over $\mathbb{R}$.

The last statement follows since two maximal toral subalgebras are conjugate by an element of H if they are the Lie algebras of maximal tori which are conjugate by an element of H. □

Definition 5.8. We say $\mathfrak{t} \subset \mathfrak{h}$ is a *toral subalgebra* of $\mathfrak{h}$ if $\mathfrak{t}$ is an abelian subalgebra consisting of semisimple elements. A *maximal toral subalgebra* of $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{h}$ that is not properly contained in another toral subalgebra of $\mathfrak{h}$.

Lemma 5.9. *If $\mathfrak{t} \subset \mathfrak{h}$ is a maximal toral subalgebra, then $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{h}_{\mathbb{C}}$ is a maximal toral subalgebra.*

Proof. Suppose $\mathfrak{t} \subset \mathfrak{h}$ is a maximal toral subalgebra with complexification $\mathfrak{t}_{\mathbb{C}}$. Note $Z_H(\mathfrak{t}) \subset Z_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$ is a real form. Choose a maximal toral subalgebra $\widetilde{\mathfrak{t}_{\mathbb{C}}} \subset \mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{h}_{\mathbb{C}}$ containing $\mathfrak{t}_{\mathbb{C}}$. Clearly $\widetilde{\mathfrak{t}_{\mathbb{C}}}$ is contained in $Z_{\mathfrak{h}_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$, the Lie algebra of $Z_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$, since $\mathfrak{t}_{\mathbb{C}}$ is contained in $\widetilde{\mathfrak{t}_{\mathbb{C}}}$ and $\widetilde{\mathfrak{t}_{\mathbb{C}}}$ is abelian. By Lemma 5.7, there exists a maximal torus $\widetilde{T_{\mathbb{C}}}$ with Lie algebra $\widetilde{\mathfrak{t}_{\mathbb{C}}}$. This maximal torus of $H_{\mathbb{C}}$ is contained in $Z_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$ since it is abelian and $\mathfrak{t}_{\mathbb{C}}$ is contained in its Lie algebra. All maximal tori in a complex, linear algebraic group are conjugate; therefore, every maximal torus in $Z_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$ is a maximal torus in $H_{\mathbb{C}}$.

Now, by 18.2 Theorem (i) on page 218 of [Bor91], there exists a maximal torus $B_{\mathbb{C}} \subset Z_{H_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$ that is defined over $\mathbb{R}$. Then its Lie algebra $\mathfrak{b}_{\mathbb{C}}$ is defined over $\mathbb{R}$ with real points $\mathfrak{b}$. Since $\mathfrak{t}_{\mathbb{C}}$ is in the center of $Z_{\mathfrak{h}_{\mathbb{C}}}(\mathfrak{t}_{\mathbb{C}})$, we must have $\mathfrak{t}_{\mathbb{C}} \subset \mathfrak{b}_{\mathbb{C}}$. In particular, $\mathfrak{t} \subset \mathfrak{b}$. By maximality, we must have $\mathfrak{t} = \mathfrak{b}$. Therefore, $\mathfrak{t}_{\mathbb{C}} = \mathfrak{b}_{\mathbb{C}}$ is a maximal toral subalgebra of $\mathfrak{h}_{\mathbb{C}}$ as desired. □

Proposition 5.10. *Let H be a Lie group with Lie algebra $\mathfrak{h}$, and suppose that there exists a real, linear algebraic group H_1 with Lie algebra $\mathfrak{h}$ as well. Assume $\mathfrak{h}_s \subset \mathfrak{h}$ is dense. Then there exists a finite number of maximal toral subalgebras $\mathfrak{t}_1, \ldots, \mathfrak{t}_r$ in $\mathfrak{h}$ such that*

$$\overline{\bigcup_{j=1}^{r} \bigcup_{g \in H} \mathrm{Ad}(g) \cdot \mathfrak{t}_j} = \mathfrak{h}.$$

Proof. Note that H_e and $(H_1)_e$ are both connected Lie groups with the same Lie algebra. In particular, we have

$$\mathrm{Ad}(H_e) = \mathrm{Ad}((H_1)_e) \subset \mathrm{Aut}(\mathfrak{h}).$$

Now, there are finitely many $((H_1)_{\mathbb{C}})_e(\mathbb{R})$ conjugacy classes of maximal toral subalgebras in $\mathfrak{h}$ by Lemma 5.7 and Lemma 5.9. Hence, there are finitely many $H_1 \supset ((H_1)_{\mathbb{C}})_e(\mathbb{R})$ conjugacy classes of maximal toral subalgebras in $\mathfrak{h}$. Also, note that H_1 has finitely many connected components since H_1 is algebraic. Hence, there are finitely many $(H_1)_e$ conjugacy classes of maximal tori in $\mathfrak{h}$. Since $\mathrm{Ad}(H_e) = \mathrm{Ad}((H_1)_e)$, there are also finitely many H conjugacy classes of maximal tori in $\mathfrak{h}$. Let $\mathfrak{t}_1, \dots, \mathfrak{t}_r$ be representatives of these conjugacy classes. Since every semisimple element in $\mathfrak{h}$ is contained in a maximal toral subalgebra, we have

$$\mathfrak{h}_s = \bigcup_{j=1}^{r} \bigcup_{g \in H} \mathrm{Ad}(g) \cdot \mathfrak{t}_j$$

for maximal toral subalgebras $\mathfrak{t}_1, \dots, \mathfrak{t}_r$. Since we assumed $\mathfrak{h}_s \subset \mathfrak{h}$ to be dense, the Proposition follows. □

5.3. **Proof of Theorem 5.3.** Recall that we fixed an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, and let $|\cdot|$ be the associated norm. Using the inner product and division by i we can identify $i\mathfrak{g}^* \simeq \mathfrak{g}$, and by abuse of notation also denote by $|\cdot|$ the corresponding norm on $i\mathfrak{g}^*$.

Recall that for $x \in X$, $G_x \subset G$ is the stabilizer subgroup of x and $\mathfrak{g}_x$ is the associated Lie algebra. Let us first show that Proposition 5.10 allows us to conjugate any semisimple element $Y \in \mathfrak{g}_x$ to an elment in $\mathfrak{h}$ in a bounded way.

Lemma 5.11. *Let G be a real, linear algebraic group, let $H \subset G$ be a closed subgroup, and suppose there exists a real algebraic subgroup $H_1 \subset G$ such that the Lie algebra of H_1, $\mathfrak{h}_1$, and the Lie algebra of H, $\mathfrak{h}$, are equal as subsets of the Lie algebra of G, $\mathfrak{g}$. Then there is a constant C such that for any $x \in X = G/H$ and any semisimple element $Y \in \mathfrak{g}_x$ there is a $g_Y \in G$ such that $\mathrm{Ad}(g_Y^{-1})Y \in \mathfrak{h}$ and*

$$|\mathrm{Ad}(g_Y^{-1})Y| \leq C|Y|.$$

Proof. For any element $Y \in \mathfrak{g}_x$ there is an element g_x such that $\mathrm{Ad}(g_x^{-1})Y \in \mathfrak{h}$. If Y is semisimple then also $\mathrm{Ad}(g_x^{-1})Y \in \mathfrak{h}$ is semisimple and as in the proof of Proposition 5.10 we may conjugate $\mathrm{Ad}(g_x^{-1})Y$ by some $h \in H$ to an element in one of the finitely many toral subalgebras $\mathfrak{t}_1, \dots, \mathfrak{t}_r$. In particular, putting $g_Y = g_x h$, we obtain

$$\mathrm{Ad}(g_Y^{-1})Y \in \mathfrak{t}_j$$

for some $j = 1, \dots, r$.

Now lets choose an embedding

$$\rho : \ G_{\mathbb{C}} \to \mathrm{GL}(N, \mathbb{C}),$$

and choose a maximal torus $T_{\mathbb{C}} \subset G_{\mathbb{C}}$. Since $\rho(T_{\mathbb{C}})$ is a group of commuting diagonalizable matrices, they are simultaneously diagonalizable. In particular, after conjugating ρ, we may assume that ρ takes $T_{\mathbb{C}}$ into the set of diagonal matrices in $\mathrm{GL}(N, \mathbb{C})$. Now, we fix a new norm $|\cdot|_\rho$ on $\mathfrak{g}_{\mathbb{C}}$ by

$$|X|_\rho := |d\rho(X)|_{\mathrm{op}}.$$

That is, the norm of X is the operator norm of the endomorphism $d\rho(X)$ of $\mathbb{C}^N$. Suppose $X \in \mathfrak{g}_{\mathbb{C}}$ and $v \in \mathbb{C}^N$ is an eigenvector for $d\rho(X)$ with eigenvalue λ. Then $d\rho(X)v = \lambda v$ and we deduce $|X|_\rho \geq |\lambda|$. More generally, we have

$$|X|_\rho \geq \sup_{\lambda \in \operatorname{Spec} d\rho(X)} |\lambda|$$

where $\operatorname{Spec} d\rho(X)$ denotes the set of eigenvalues of $d\rho(X)$. On the other hand, if $X \in \mathfrak{t}_{\mathbb{C}} = \operatorname{Lie}(T_{\mathbb{C}})$, then $d\rho(X)$ is a diagonal matrix, and the above inequality is in fact an equality. If $g \in G_{\mathbb{C}}$, since $d\rho(\operatorname{Ad}(g)X) = \rho(g)d\rho(X)\rho(g)^{-1}$ is conjugate to $d\rho(X)$, we deduce

$$|X|_\rho = \sup_{\lambda \in \operatorname{Spec} d\rho(X)} |\lambda| = \sup_{\lambda \in \operatorname{Spec} d\rho(\operatorname{Ad}_g X)} |\lambda| \leq |\operatorname{Ad}(g)X|_\rho.$$

In particular, for every semisimple $G_{\mathbb{C}}$ orbit $\mathcal{O} \subset \mathfrak{g}_{\mathbb{C}}$, the norm $|\cdot|_\rho$ takes its minimum value on $\mathfrak{t}_{\mathbb{C}} \cap \mathcal{O}$ (note that the latter set is not empty since all maximal tori are conjugate and every semisimple element belongs to some maximal torus). Recall that $\mathfrak{t}_j \subset \mathfrak{h}$ are maximal toral subalgebras so accroding to Lemma 5.9, $\mathfrak{t}_{j,\mathbb{C}} \subset \mathfrak{h}_{\mathbb{C}}$ is a maximal toral subalgebra. Applying Lemma 5.7 to $G_{\mathbb{C}}$ and using $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ we fix for each j a maximal torus $T_{j,\mathbb{C}} \subset G_{\mathbb{C}}$ such that

$$\operatorname{Lie}(T_{j,\mathbb{C}}) \supset \mathfrak{t}_{j,\mathbb{C}}$$

For each j, we fix a homomorphism

$$\rho_j \colon G_{\mathbb{C}} \to \mathrm{GL}(N_j, \mathbb{C})$$

for which $\rho_j(T_{j,\mathbb{C}})$ is a collection of diagonal matrices. By this map we obtain a finite number of norms $|\ |_{\rho_j}$ on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and thus also in the real subspace $\mathfrak{g}$. Since all norms on a finite dimensional vectorspace are equivalent we obtain a constant $d > 0$ such that for all $j = 1, \ldots, r$

$$\frac{1}{d}|X| \leq |X|_{\rho_j} \leq d|X|. \tag{5.1}$$

Now suppose that j is such that $\operatorname{Ad}(g_Y^{-1})Y \in \mathfrak{t}_j$. As noted before, when restricted to the orbit $G_{\mathbb{C}} \cdot Y$, the norm $|\cdot|_{\rho_j}$ takes its minimum on $\mathfrak{t}_{j,\mathbb{C}}$. Therefore,

$$|\operatorname{Ad}(g_Y^{-1})Y|_{\rho_j} \leq |Y|_{\rho_j}$$

and using (5.1) we get

$$|\operatorname{Ad}(g_Y^{-1})Y| \leq d^2|Y|.$$

Note that the constant d^2 did only depend on the choices of the norms ρ_j and is thus independent of Y and g_Y and we have thus proven Lemma 5.11. $\square$

As a corollary of this lemma we can now prove (3.8) and (3.10) for $U_{\mathrm{CU,WF}} = U_{\mathrm{CU,SS}} = U_{\mathrm{CU}} \subset \mathfrak{g}$ being any precompact open neighborhood of $0 \in \mathfrak{g}$: Let $g_x \in G$ be an arbitrary representative of $x = g_xH \in X$. After choosing a nonzero point of reference in $\mathcal{D}_x^{1/2}$ and using (3.1) and (3.3) we can identify $\mathcal{D}_x^{1/2} \cong \mathbb{C}$ and

$$\sigma_x(e^Y) \cong |\det_{T_{eH}X}(de^{\operatorname{Ad}(g_x^{-1})Y}{}_{|eH})|^{-1/2}$$

for any $Y \in \mathfrak{g}_x$. From continuity we conclude

$$|\det_{T_{eH}X}(de^{\operatorname{Ad}(g_x^{-1})Y}{}_{|eH})|^{-1/2} \leq C_1|\operatorname{Ad}(g_x^{-1})Y|.$$

Note that the left side is independent of the choice of the representative g_x. Now suppose that $Y \in \mathfrak{g}_x$ is semisimple, then we can take $g_x = g_Y$ according to Lemma 5.11 and we obtain

$$|\det_{T_{eH}X}(de^{\mathrm{Ad}(g_Y^{-1})Y}{}_{|eH})|^{-1/2} \leq C_1 C|Y|.$$

By the precompactnes of U_{CU}, we have thus established (3.8) and (3.10) on the dense subset of semisimple elements and finally by continuity (3.8) and (3.10) follow on the whole set U_{CU}.

In order to prove the central estimates (3.7) and (3.9), we will perform a partial integration with respect to some vectors $Y_x \in \mathfrak{g}_x$ for every $x \in X$ satisfying certain properties. We first show that we can choose them appropriately.

As we assume in Theorem 5.3 that the representation τ is finite dimensional, we know that $\mathrm{WF}(\tau_x) = \mathrm{SS}(\tau_x) = \{0\}$, so

$$\mathcal{W} = \mathcal{S} = \overline{\bigcup_{x\in X} q_x^{-1}(0)} = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}$$

where $(\mathfrak{g}/\mathfrak{g}_x)^*$ are those linear functionals on $\mathfrak{g}$ that vanish on $\mathfrak{g}_x$.

Lemma 5.12. *Suppose $\eta_0 \notin \mathcal{W} = \mathcal{S}$. Then there exists an open neighborhood $\eta_0 \in \Omega \subset i\mathfrak{g}^* \setminus \mathcal{W}$ such that the nonnegative constant*

$$C_\Omega := \inf_{x\in X} \sup_{\substack{Y_x \in \mathfrak{g}_x \\ |Y_x|=1}} \inf_{\xi\in\Omega} |\langle \xi, Y_x\rangle|$$

is nonzero.

Proof. We prove the Lemma by contradiction. Let $\{\Omega_m\}$ be a sequence of open sets in $i\mathfrak{g}^*$ such that $\Omega_m \supset \Omega_{m+1}$ for every m and

$$\bigcap_m \Omega_m = \{\eta_0\}.$$

Suppose $C_{\Omega_m} = 0$ for every m. Then, for every m, we must be able to find a sequence $\{x_n^m\}$ of elements of X such that

$$c_n^m = \sup_{\substack{Y_{n,m} \in \mathfrak{g}_{x_n^m} \\ |Y_{n,m}|=1}} \inf_{\xi\in\Omega_m} |\langle \xi, Y_{n,m}\rangle|$$

converges to zero as $n \to \infty$. Choose a sequence of increasing natural numbers $\{n_m\}$ such that $c_{n_m}^m$ converges monotonically to zero as $m \to \infty$.

Since the Grassmannian of all vector spaces of dimension $\dim \mathfrak{h}$ in $\mathfrak{g}$ is a compact space, after passing to a subsequence, we may assume that $\{\mathfrak{g}_{x_{n_m}^m}\} \to V$ where $V \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ of dimension $\dim \mathfrak{h}$.

Next, we will show $q_V(\eta_0) = 0$. If $Y \in V$ and $|Y| = 1$, then we may write $Y = \lim_{m\to\infty} Y_m$ with $Y_m \in \mathfrak{g}_{x_{n_m}^m}$ and $|Y_m| = 1$. We note

$$\inf_{\xi\in\Omega_m} |\langle \xi, Y_m\rangle| \leq c_{n_m}^m.$$

For every $\xi \in \Omega_m$ we can bound

$$|\langle \eta_0, Y\rangle| \leq |\langle \eta_0, Y - Y_m\rangle| + |\langle \eta_0 - \xi, Y_m\rangle| + |\langle \xi, Y_m\rangle|.$$

By choosing ξ such that the last term becomes as small as possible, we obtain

$$|\langle \eta_0, Y\rangle| \leq |\langle \eta_0, Y - Y_m\rangle| + \sup_{\xi\in\Omega_m} |\langle \eta_0 - \xi, Y_m\rangle| + \inf_{\xi\in\Omega_m} |\langle \xi, Y_m\rangle|.$$

The right hand side converges to zero as $m \to \infty$ since $c^m_{n_m} \to 0$ and $\cap \Omega_m = \eta_0$. Therefore, η_0 vanishes on V.

Finally, we show that

$$i(\mathfrak{g}/V)^* \subset \overline{\bigcup_{x \in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}.$$

Indeed, it is not difficult to see that if $\{\mathfrak{g}_{x_n}\} \to V$ in the Grassmannian of dimension $\dim \mathfrak{h}$ subspaces of $\mathfrak{g}$, then $\{\mathfrak{g}_{x_n}^\perp\} \to V^\perp$ in the Grassmannian of dimension $\dim \mathfrak{g} - \dim \mathfrak{h}$ subspaces of $\mathfrak{g}$. Dividing by i and utilizing our fixed inner product on $\mathfrak{g}$, we may identify imaginary valued linear functionals on $\mathfrak{g}$ that vanish on V (resp. $\mathfrak{g}_{x_n}$) with $V^\perp$ (resp. $\mathfrak{g}_{x_n}^\perp$). The statement now follows.

Now, $\eta_0 \in i(\mathfrak{g}/V)^*$. Hence, $\eta_0 \in \overline{\bigcup_{x \in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}$. But, this contradicts our hypothesis. Hence, $C_{\Omega_m} \neq 0$ for some m, and the Lemma has been proven. □

Now putting together Lemma 5.11 and Lemma 5.12 we can specify the vectors Y_x.

Corollary 5.13. *Let G be a real, linear algebraic group, let $H \subset G$ be a closed subgroup, and suppose there exists a real algebraic group $H_1 \subset G$ such that the Lie algebra of H_1, $\mathfrak{h}_1$, and the Lie algebra of H, $\mathfrak{h}$, are equal as subsets of the Lie algebra of G, $\mathfrak{g}$. In addition, assume that the set of semisimple elements of $\mathfrak{h}$, denoted $\mathfrak{h}_s$ is dense in $\mathfrak{h}$. Fix*

$$\eta_0 \notin \mathcal{W} = \mathcal{S} = \overline{\bigcup_{x \in X} i(\mathfrak{g}/\mathfrak{g}_x)^*},$$

and let $\eta_0 \in \Omega \subset i\mathfrak{g}^*$ be the open set from Lemma 5.12. For every $x \in X$, the set we can choose Y_x and g_x satisfying

1. *$Y_x \in \mathfrak{g}_x$ for all $x \in X$*
2. *$|Y_x| = 1$ for all $x \in X$*
3. *For all $x \in X$, we have the inequality*

$$\inf_{\xi \in \Omega} |\langle \xi, Y_x \rangle| > \frac{C_\Omega}{2}$$

4. *For each $x \in X$, there exists $g_x \in G$ such that $\mathrm{Ad}(g_x^{-1})Y_x \in \mathfrak{h}$ and*

$$|\mathrm{Ad}(g_x^{-1})Y_x| \le C$$

uniformly in x.

Proof. From Lemma 5.12 we conclude that the set of possible choices of Y_x satisfying (1)-(3)

$$\left\{ Y \in \mathfrak{g}_x \,\middle|\, \inf_{\xi \in \Omega} |\langle \xi, Y \rangle| > \frac{C_\Omega}{2} \text{ and } |Y| = 1 \right\}$$

is an open subset of the unit sphere in $\mathfrak{g}_x$.

As $\mathfrak{h}_s \subset \mathfrak{h}$ is dense we can choose Y_x semisimple. We may then choose $g_x = g_Y$ according to Lemma 5.11 and obtain (4). □

For the remainder of the proof of Theorem 5.3, we fix choices of Y_x and g_x satisfying the above four properties.

We are now ready to prove the central estimates (3.7) and (3.9) by partial integration with respect to Y_x. We define the differential operator $L_x : C^\infty(\mathfrak{g}_x) \to C^\infty(\mathfrak{g}_x)$

by its action

$$L_x\varphi(Y) := \frac{\partial_{Y_x}}{t\langle\xi, Y_x\rangle}\varphi(Y) = \frac{1}{t\langle\xi, Y_x\rangle}\frac{d}{ds}_{|s=0}\left(\varphi(Y+sY_x)\right)$$

and note that we have for any $k\in\mathbb{N}$ the identity

$$L_x^k\left(e^{\langle t\xi, Y\rangle}\right) = e^{\langle t\xi, Y\rangle}.$$

Thus we can insert this differential operator on the left hand side of (3.7) and obtain

$$\begin{aligned}
&\left|\int_{\mathfrak{g}_x}\langle\tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x}\cdot\left((\sigma_x(e^Y)z_1)\otimes\overline{z_2}\right)\cdot\varphi(Y)L_x^N e^{\langle t\xi, Y\rangle}dY\right|_{\mathcal{D}_x^1}\\
&=\left|\frac{1}{t\langle\xi, Y_x\rangle}\right|^N\left|\sum_{q+r+s=N}C_{q,r,s}\int_{\mathfrak{g}_x}\langle\partial_{Y_x}^q\tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x}\right.\\
&\qquad\qquad\left.\cdot\left((\partial_{Y_x}^r\sigma_x(e^Y)z_1)\otimes\overline{z_2}\right)\cdot\partial_{Y_x}^s\varphi(Y)e^{\langle t\xi, Y\rangle}dY\right|_{\mathcal{D}_x^1}\\
&=\left|\frac{1}{t\langle\xi, Y_x\rangle}\right|^N\left|\sum_{q+r+s=N}C_{q,r,s}\int_{\mathfrak{g}_x}\langle\tau_x(e^Y)d\tau_x(Y_x)^q v_1, v_2\rangle_{\mathcal{V}_x}\right.\\
&\qquad\qquad\left.\cdot\left((\sigma_x(e^Y)d\sigma_x(Y_x)^r z_1)\otimes\overline{z_2}\right)\cdot\partial_{Y_x}^s\varphi(Y)e^{\langle t\xi, Y\rangle}dY\right|_{\mathcal{D}_x^1}
\end{aligned}$$

We obtain an analogous formula for the singular spectrum case if we insert this differential operator on the left hand side of (3.9). In the singular spectrum case formula, φ is replaced by φ_{N,U_1,U_2}.

Back in the wavefront case, for any $\varphi\in C_c^\infty(U_{\mathrm{CU}})$ the precompactnes of U_{CU} assures a bound of the supremum norm

$$\|\partial_{Y_x}^s\varphi(Y)\|_\infty\le C_{s,\varphi}$$

uniformly in $x\in X$. For the singular spectrum case, part (3) of the definition of the family of functions φ_{N,U_1,U_2} given directly above Definition 3.4 implies the stronger bounds

$$\|\partial_{Y_x}^s\varphi_{N,U_1,U_2}(Y)\|_\infty\le C_{U_1,U_2}^{s+1}(N+1)^s$$

in the singular spectrum case. Additionally we have shown above that $|\sigma_x(e^Y)|_{\mathrm{op}}<C$ uniformly in $x\in X$, $Y\in U_{\mathrm{CU}}\cap\mathfrak{g}_x$ and a uniform bound for $|\tau_x(e^Y)|_{\mathrm{op}}$ follows trivially from the unitarity of τ_x. It thus remains to prove uniform bounds for $|d\tau_x(Y_x)|_{\mathrm{op}}$ and $|d\sigma_x(Y_x)|_{\mathrm{op}}$. Using once more (3.2) we obtain

$$|d\tau_x(Y_x)|_{\mathrm{op}} = |d\tau(\mathrm{Ad}(g_x^{-1})Y_x)|_{\mathrm{op}}.$$

Now the continuity of $d\tau:\mathfrak{h}\to\mathrm{End}(V)$ and $|\ |_{\mathrm{op}}:\mathrm{End}(V)\to\mathbb{R}$ imply

$$|d\tau_x(Y_x)|_{\mathrm{op}}\le C|\mathrm{Ad}(g_x^{-1})Y_x|\le\tilde{C}$$

where the last inequality is justified by Lemma 5.11. The same arguments apply to $|d\sigma_x(Y_x)|_{\mathrm{op}}$. Putting everything together, in the wavefront case, we obtain

$$\left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot \big((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\big) \cdot \varphi(Y) L_x^N e^{\langle t\xi, Y\rangle} dY \right|_{\mathcal{D}_x^1}$$
$$\leq \left| \frac{1}{t\langle \xi, Y_x\rangle} \right|^N C_{N,\varphi}(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}}) \otimes (\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}}).$$

Analogously, in the singular spectrum case, we obtain

$$\left| \int_{\mathfrak{g}_x} \langle \tau_x(e^Y)v_1, v_2\rangle_{\mathcal{V}_x} \cdot \big((\sigma_x(e^Y)z_1) \otimes \overline{z_2}\big) \cdot \varphi_{N,U_1,U_2}(Y) L_x^N e^{\langle t\xi, Y\rangle} dY \right|_{\mathcal{D}_x^1}$$
$$\leq \left| \frac{1}{t\langle \xi, Y_x\rangle} \right|^N C_{U_1,U_2}^{N+1}(N+1)^N(\|v_1\|_{\mathcal{V}_x}|z_1|_{\mathcal{D}_x^{1/2}}) \otimes (\|v_2\|_{\mathcal{V}_x}|z_2|_{\mathcal{D}_x^{1/2}}).$$

Finally from property (3) of Corollary 5.13 we obtain that $|1/\langle \xi, Y_x\rangle|$ is uniformly bounded for $x \in X$ and $\xi \in \Omega$. This finishes the proofs of (3.7) and (3.9) and thus of Theorem 5.3.

6. Applications and Examples

In this section, we consider applications of Theorem 1.1 and Theorem 1.2, and we consider examples of those applications. We begin by expanding upon Corollary 1.4. As in the introduction, let G_x be the stabilizer in G of $x \in X$, and let $\mathfrak{g}_x$ denote the Lie algebra of G_x. For each $x \in X$, we identify iT_x^*X with $i(\mathfrak{g}/\mathfrak{g}_x)^*$ in the obvious way.

Corollary 6.1. *Suppose G is a real, reductive algebraic group, and suppose X is a homogeneous space for G with an invariant measure. Then*

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma \in \operatorname{supp} L^2(X) \\ \sigma \in \widehat{G}_{temp}}} \mathcal{O}_\sigma \right) \subset \overline{\bigcup_{x\in X} iT_x^*X} = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}. \tag{6.1}$$

Intersecting with the set of regular semisimple elements, we obtain the equality

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma \in \operatorname{supp} L^2(X) \\ \sigma \in \widehat{G}'_{temp}}} \mathcal{O}_\sigma \right) \cap (i\mathfrak{g}^*)' = \overline{\bigcup_{x\in X} iT_x^*X} \cap (i\mathfrak{g}^*)' = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*} \cap (i\mathfrak{g}^*)'. \tag{6.2}$$

If, in addition, $\operatorname{supp} L^2(X) \subset \widehat{G}_{temp}$*, then we obtain equality without intersecting with the set of regular semisimple elements,*

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma \in \operatorname{supp} L^2(X) \\ \sigma \in \widehat{G}_{temp}}} \mathcal{O}_\sigma \right) = \overline{\bigcup_{x\in X} iT_x^*X} = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}. \tag{6.3}$$

This Corollary is a consequence of Theorems 1.1 and 1.2 of [HHO], Theorem 1.1 of [Har], and Theorem 1.1 of this paper. The last statement is especially interesting to us because of the recent work of Benoist and Kobayashi [BK], which gives a large class of homogeneous spaces X for a real, reductive algebraic group G for which $\operatorname{supp} L^2(X) \subset \widehat{G}_{\mathrm{temp}}$. Let us write down a brief proof of Corollary 6.1.

Proof. Consider the representation $L^2(X)$ of G. We may decompose this representation into irreducibles as

$$L^2(X) \cong \int_{\sigma\in\widehat{G}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu$$

where μ is a nonnegative measure on the unitary dual $\widehat{G}$. We may decompose

$$\widehat{G} = \widehat{G}_{\mathrm{temp}} \bigcup \left(\widehat{G} - \widehat{G}_{\mathrm{temp}}\right)$$

as the union of a closed set and an open set and we may analogously decompose

$$\mu = \mu|_{\widehat{G}_{\mathrm{temp}}} + \mu|_{\widehat{G}-\widehat{G}_{\mathrm{temp}}}.$$

We then obtain an analogous direct sum decomposition of the representation $L^2(X)$,

$$L^2(X) \cong \int_{\sigma\in\widehat{G}_{\mathrm{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}_{\mathrm{temp}}} \bigoplus \int_{\sigma\in\widehat{G}-\widehat{G}_{\mathrm{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}-\widehat{G}_{\mathrm{temp}}}.$$

Since every matrix coefficient of the first summand is also a matrix coefficient of $L^2(X)$, we obtain the inclusion

$$\mathrm{WF}(L^2(X)) \supset \mathrm{WF}\left(\int_{\sigma\in\widehat{G}_{\mathrm{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}_{\mathrm{temp}}}\right).$$

Now, by Theorem 1.2 of [HHO], we obtain

$$\mathrm{WF}\left(\int_{\sigma\in\widehat{G}_{\mathrm{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}_{\mathrm{temp}}}\right) = \mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}_{\mathrm{temp}}}} \mathcal{O}_\sigma\right).$$

On the other hand, one obtains from Theorem 1.1 of this paper

$$\mathrm{WF}(L^2(X)) = \overline{\bigcup_{x\in X} iT_x^*X} = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}.$$

Now, statement (6.1) follows. If, in addition, $\operatorname{supp} L^2(X) \subset \widehat{G}_{\mathrm{temp}}$, then

$$\mu|_{\widehat{G}-\widehat{G}_{\mathrm{temp}}} = 0$$

and we obtain

$$\mathrm{WF}(L^2(X)) = \mathrm{WF}\left(\int_{\sigma\in\widehat{G}_{\mathrm{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}_{\mathrm{temp}}}\right).$$

Now, statement (6.3) follows from Theorem 1.2 of [HHO] together with Theorem 1.1 of this paper (which utilizes Theorem 1.1 of [HHO] in its proof).

To show statement (6.2), we require a different decomposition of measures. We instead break up

$$\widehat{G} = \widehat{G}'_{\mathrm{temp}} \bigcup \left(\widehat{G} - \widehat{G}'_{\mathrm{temp}}\right)$$

into the union of an open set and a closed set and we have the corresponding decomposition of measures

$$\mu = \mu|_{\widehat{G}'_{\text{temp}}} + \mu|_{\widehat{G}-\widehat{G}'_{\text{temp}}}.$$

This yields a decomposition of representations

$$L^2(X) \cong \int_{\sigma\in\widehat{G}'_{\text{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}'_{\text{temp}}} \bigoplus \int_{\sigma\in\widehat{G}-\widehat{G}'_{\text{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}-\widehat{G}'_{\text{temp}}}.$$

Now, as shown in Proposition 1.3 of [How81], every matrix coefficient of $L^2(X)$ decomposes into the sum of a matrix coefficient of the first representation plus a matrix coefficient of the second representation which means that we can write $\mathrm{WF}(L^2(X))$ as a union of

$$\mathrm{WF}\left(\int_{\sigma\in\widehat{G}'_{\text{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}'_{\text{temp}}}\right)$$

and

$$\mathrm{WF}\left(\int_{\sigma\in\widehat{G}-\widehat{G}'_{\text{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}-\widehat{G}'_{\text{temp}}}\right).$$

Now, Theorem 1.1 of [Har] says that the second set is contained in the singular set, $i\mathfrak{g}^* - (i\mathfrak{g}^*)'$, since it is a direct integral of singular representations. Therefore, we obtain

$$\mathrm{WF}(L^2(X)) \cap (i\mathfrak{g}^*)' = \mathrm{WF}\left(\int_{\sigma\in\widehat{G}'_{\text{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}'_{\text{temp}}}\right) \cap (i\mathfrak{g}^*)'$$

By Theorem 1.1 of this paper (which utilizes Theorem 1.1 of [HHO] in its proof)

$$\mathrm{WF}(L^2(X)) = \overline{\bigcup_{x\in X} iT_x^*X} = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}.$$

And by Theorem 1.2 of [HHO], we obtain

$$\mathrm{WF}\left(\int_{\sigma\in\widehat{G}'_{\text{temp}}}^{\oplus} \sigma^{\oplus m(L^2(X),\sigma)} d\mu|_{\widehat{G}'_{\text{temp}}}\right) = \mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{\text{temp}}}} \mathcal{O}_\sigma\right).$$

Statement (6.2) now follows. □

Utilizing Theorem 1.2, we give a variant of Corollary 1.4 for finite rank vector bundles.

Corollary 6.2. *Suppose G is a real, reductive algebraic group, suppose X is a homogeneous space for G, let $\mathcal{D}^{1/2} \to X$ denote the bundle of complex half densities on X (see Appendix A), and suppose $\mathcal{V} \to X$ is a finite rank, G equivariant, Hermitian vector bundle on X. Assume that for some (equivalently any) $x \in X$, there is a closed, real, linear algebraic subgroup $\widetilde{G_x} \subset G$ whose Lie algebra is $\mathfrak{g}_x$ (of*

course, this is satisfied if G_x is itself an algebraic subgroup of G) and $\mathfrak{g}_x$ contains a dense subset of semisimple elements. Then

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X,\mathcal{D}^{1/2}\otimes\mathcal{V})\\ \sigma\in\widehat{G}_{temp}}} \mathcal{O}_\sigma\right) \subset \overline{\bigcup_{x\in X} iT_x^*X} = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}. \tag{6.4}$$

Intersecting with the set of regular semisimple elements, we obtain the equality

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X,\mathcal{D}^{1/2}\otimes\mathcal{V})\\ \sigma\in\widehat{G}'_{temp}}} \mathcal{O}_\sigma\right) \cap (i\mathfrak{g}^*)' = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*} \cap (i\mathfrak{g}^*)'. \tag{6.5}$$

If, in addition, $\operatorname{supp} L^2(X,\mathcal{D}^{1/2}\otimes\mathcal{V}) \subset \widehat{G}_{temp}$*, then we obtain equality without intersecting with the set of regular, semisimple elements,*

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X,\mathcal{D}^{1/2}\otimes\mathcal{V})\\ \sigma\in\widehat{G}_{temp}}} \mathcal{O}_\sigma\right) = \overline{\bigcup_{x\in X} iT_x^*X} = \overline{\bigcup_{x\in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}. \tag{6.6}$$

This Corollary is a consequence of Theorems 1.1 and 1.2 of [HHO], Theorem 1.1 of [Har], and Theorem 1.2 of this paper. The verification of Corollary 6.2 is nearly identical to the verification of Corollary 6.1 with Theorem 1.1 of this paper replaced by Theorem 1.2 of this paper in all arguments. Utilizing results of Matumoto [Mat92], we will point out in Section 7 that all three of the statements of Corollary 6.2 may fail if one does not assume that $\mathfrak{g}_x$ contains a dense subset of semisimple elements.

In order to utilize Corollary 6.1 and Corollary 6.2, it is useful to write these statements in terms of purely imaginary valued linear functionals on the Cartan subalgebras of $\mathfrak{g}$, the Lie algebra of G. Suppose $\mathfrak{b}\subset\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$, and identify $i\mathfrak{b}^*\subset i\mathfrak{g}^*$ utilizing the decomposition

$$\mathfrak{g} = \mathfrak{b}\oplus[\mathfrak{g},\mathfrak{b}].$$

Given σ, an irreducible, tempered representation of G with regular infinitesimal character, define

$$\lambda_{\sigma,\mathfrak{b}} := \mathcal{O}_\sigma \cap i\mathfrak{b}^*.$$

When $\mathfrak{b}$ is understood, we will often just write λ_σ. If σ is an irreducible, tempered representation with regular infinitesimal character, then λ_σ is a single W orbit for the real Weyl group $W = W(G,B) = N_G(B)/B$. Here $B := N_G(\mathfrak{b}) \subset G$ is the Cartan subgroup of G with Lie algebra $\mathfrak{b}$.

Corollary 6.3. *Let G be a real, reductive algebraic group, and let X be a homogeneous space for G with a nonzero invariant density. Suppose $\mathfrak{b}\subset\mathfrak{g}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. As before, G_x is the stabilizer in G of $x\in X$*

and $\mathfrak{g}_x$ is the Lie algebra of G_x. In addition, $(i\mathfrak{b}^*)' = i\mathfrak{b}^* \cap (i\mathfrak{g}^*)'$. Then

$$\text{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{temp}}} \lambda_{\sigma,\mathfrak{b}}\right) \cap (i\mathfrak{b}^*)' = \overline{\{\xi\in i\mathfrak{b}^* | \exists\ x\in X s.t.\ \ \xi|_{\mathfrak{g}_x} = 0\ \}} \cap (i\mathfrak{b}^*)'. \tag{6.7}$$

The asymptotic cone is taken inside the vector space $i\mathfrak{b}^*$.

Proof. We show how to deduce Corollary 6.3 from Corollary 6.1. To show that the right hand side is contained in the left hand side, take $\xi \in (i\mathfrak{b}^*)'$ with $\xi|_{\mathfrak{g}_x} = 0$ for some $x \in X$, and choose an open cone $\xi \in \mathcal{C}_1 \subset (i\mathfrak{b})^*$ for which $\mathcal{C}_1 \subset (i\mathfrak{b}^*)'$. Choose a precompact, open subset $e \in K \subset G$, and consider the open cone $\xi \in \mathcal{C} := K \cdot \mathcal{C}_1 \subset i\mathfrak{g}^*$. Then by Corollary 6.1, we deduce

$$\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{\text{temp}}}} \mathcal{O}_\sigma\right) \cap \mathcal{C}$$

is unbounded. Now, we have a proper, continuous map $K \times \mathcal{C}_1 \to i\mathfrak{g}^*$, and we know that the image of the set

$$K \times \left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{\text{temp}}}} \lambda_{\sigma,\mathfrak{b}}\right) \cap \mathcal{C}_1$$

is unbounded in $i\mathfrak{g}^*$. We deduce that

$$\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{\text{temp}}}} \lambda_{\sigma,\mathfrak{b}}\right) \cap \mathcal{C}_1$$

is unbounded in $i\mathfrak{b}^*$. Therefore,

$$\xi \in \text{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{\text{temp}}}} \lambda_{\sigma,\mathfrak{b}}\right).$$

If $\xi \in (i\mathfrak{b}^*)'$ is a limit of $\xi_n \in (i\mathfrak{b}^*)'$ with $\xi_n|_{\mathfrak{g}_{x_n}} = 0$ each n, then every open cone containing ξ must also contain some ξ_n. Hence, the required set must intersect this cone in an unbounded set. We have shown that the right hand side of (6.7) is contained in the left hand side of (6.7).

Next, we show that the left hand side of (6.7) is contained in the right hand side of (6.7). Suppose

$$\xi \in \text{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X)\\ \sigma\in\widehat{G}'_{\text{temp}}}} \lambda_{\sigma,\mathfrak{b}}\right) \cap (i\mathfrak{b}^*)',$$

and let $\xi \in \mathcal{C} \subset i\mathfrak{g}^*$ be an open cone in $i\mathfrak{g}^*$ containing ξ. Without loss of generality, we may assume $\mathcal{C} \subset (i\mathfrak{g}^*)'$. Then $\mathcal{C}_1 := \mathcal{C} \cap i\mathfrak{b}^*$ is an open cone in $i\mathfrak{b}^*$ containing ξ. By our assumption on ξ, we conclude that

$$\mathcal{C}_1 \cap \mathrm{AC}\left(\bigcup_{\substack{\sigma \in \operatorname{supp} L^2(X) \\ \sigma \in \widehat{G}'_{\mathrm{temp}}}} \lambda_{\sigma,\mathfrak{b}}\right)$$

is unbounded. Then we deduce that

$$\mathcal{C} \cap \left(\bigcup_{\substack{\sigma \in \operatorname{supp} L^2(X) \\ \sigma \in \widehat{G}'_{\mathrm{temp}}}} \mathcal{O}_\sigma\right)$$

is unbounded. Therefore,

$$\xi \in \mathrm{AC}\left(\bigcup_{\substack{\sigma \in \operatorname{supp} L^2(X) \\ \sigma \in \widehat{G}'_{\mathrm{temp}}}} \mathcal{O}_\sigma\right).$$

And by Corollary 6.1, we deduce that

$$\xi \in \overline{\bigcup_{x \in X} i(\mathfrak{g}/\mathfrak{g}_x)^*}.$$

Therefore, we can write $\xi = \lim_{n\to\infty} \xi'_n$ with $\xi'_n|_{\mathfrak{g}_{x'_n}} = 0$ for some $x'_n \in X$. Using the finite to one, local diffeomorphism onto its image $G/B \times (i\mathfrak{b}^*)' \to (i\mathfrak{g}^*)'$, we deduce that we may find $\xi_n \in (i\mathfrak{b}^*)'$ such that $\xi'_n = \mathrm{Ad}^*_{g_n} \xi_n$ for some $g_n \in G$ and $\{\xi_n\} \to \xi$. If $\mathrm{Ad}_{g_n} x'_n = x_n$, then we note that

$$\xi_n|_{\mathfrak{g}_{x_n}} = 0$$

for all n, and we conclude

$$\xi \in \overline{\{\eta \in i\mathfrak{b}^* |\ \eta|_{\mathfrak{g}_x} = 0 \text{ some } x \in X\}} \cap (i\mathfrak{b}^*)'.$$

The Corollary has been verified. □

Next, we give a version of Corollary 6.3 for vector bundles.

Corollary 6.4. *Suppose G is a real, reductive algebraic group, suppose X is a homogeneous space for G, let $\mathcal{D}^{1/2} \to X$ denote the bundle of complex half densities on X (see Appendix A), and suppose $\mathcal{V} \to X$ is a finite rank, G equivariant, Hermitian vector bundle on X. As before, G_x is the stabilizer in G of $x \in X$ and $\mathfrak{g}_x$ is the Lie algebra of G_x. Assume that for some (equivalently any) $x \in X$, there is a closed, real, linear algebraic subgroup $\widetilde{G_x} \subset G$ whose Lie algebra is $\mathfrak{g}_x$ (of course, this is satisfied if G_x is itself an algebraic subgroup of G) and $\mathfrak{g}_x$ contains a dense subset of semisimple elements. Suppose $\mathfrak{b} \subset \mathfrak{g}$ is a Cartan subalgebra of the*

Lie algebra $\mathfrak{g}$ of G, and let $(i\mathfrak{b}^*)' := i\mathfrak{b}^* \cap (i\mathfrak{g}^*)'$ be the set of regular elements in $i\mathfrak{b}^*$. Then

(6.8)
$$\mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(X,\mathcal{D}^{1/2}\otimes\mathcal{V})\\ \sigma\in\widehat{G}'_{temp}}} \lambda_{\sigma,\mathfrak{b}}\right) \cap (i\mathfrak{b}^*)' = \overline{\{\xi\in i\mathfrak{b}^* | \exists\ x\in X\ s.t.\ \xi|_{\mathfrak{g}_x}=0\}} \cap (i\mathfrak{b}^*)'.$$

The asymptotic cone is taken inside the vector space $i\mathfrak{b}^*$.

The proof of Corollary 6.4 is identical to the proof of Corollary 6.3. Of course, one utilizes Corollary 6.2 instead of Corollary 6.1 in the proof.

Before doing several examples, we briefly recall some key facts about discrete series representations. If G is a real, reductive algebraic group, then G has at most one (up to G conjugacy) compact Cartan subgroup $T \subset G$. A *discrete series* representation of G is an irreducible, unitary representation σ with

$$\mathrm{Hom}_G(\sigma, L^2(G)) \neq \{0\}.$$

The group G has discrete series representations if, and only if G has a compact Cartan subgroup $T \subset G$ [HC65], [HC66], [HC70], [HC76]. Since discrete series occur discretely in the unitary dual $\widehat{G}$, whenever π is a unitary representation of G and $\sigma \in \operatorname{supp}\pi$, we must have

$$\mathrm{Hom}_G(\sigma, \pi) \neq \{0\}.$$

Cartan subalgebras $\mathfrak{t} \subset \mathfrak{g}$ which are Lie algebras of compact Cartan subgroups are called *fundamental Cartan subaglebras.* The *regular elliptic* elements in $\mathfrak{g}$ (resp. $i\mathfrak{g}^*$) are precisely the regular, semisimple elements of $\mathfrak{g}$ (resp. $i\mathfrak{g}^*$) meeting $\mathfrak{t}$ (resp. $i\mathfrak{t}^*$). Further, if σ is an irreducible, tempered representation of G, then $\mathcal{O}_\sigma \cap (i\mathfrak{t}^*)' \neq \{0\}$ if and only if σ is a discrete series representation of G.

Example 6.5. Let us consider the class of examples $G = \mathrm{Sp}(2n, \mathbb{R})$ and $H = \mathrm{Sp}(2m, \mathbb{R}) \times \mathrm{Sp}(2n-2m, \mathbb{Z})$ that is mentioned in the introduction. Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R})$ and $\mathfrak{h} = \mathrm{sp}(2m, \mathbb{R})$ denote the Lie algebras of G and H. Viewing $\mathfrak{g}$ as a set of $2n$ by $2n$ matrices and $\mathfrak{h}$ as a subset of $\mathfrak{g}$ of matrices with at least $4n^2 - 4m^2$ zeroes, one can define the complementary subspace $\mathfrak{q}$ consisting of matrices having zeroes in precisely the entries where elements of $\mathfrak{h}$ can have nonzero entries. One notes that $\mathfrak{q}$ is orthogonal to $\mathfrak{h}$ under the Killing form, B, of $\mathfrak{g}$. In particular, under the isomorphism $i\mathfrak{g}^* \cong \mathfrak{g}$, which involves dividing by i and using the Killing form, one obtains

$$i(\mathfrak{g}/\mathfrak{h})^* \cong \mathfrak{q}.$$

In particular, utilizing Corollary 6.3, we see that if there exists $\xi \in \mathfrak{q}$ that is a regular elliptic element of $\mathfrak{g}$, then there must exist infinitely many discrete series representations σ of $G = \mathrm{Sp}(2n, \mathbb{R})$ such that

$$\mathrm{Hom}_{\mathrm{Sp}(2n,\mathbb{R})}(\sigma, L^2(\mathrm{Sp}(2n, \mathbb{R})/\,\mathrm{Sp}(2m, \mathbb{R}) \times \mathrm{Sp}(2n-2m, \mathbb{Z}))) \neq \{0\}.$$

Utilizing a bit of linear algebra, one readily checks that this is the case precisely when $2m \leq n$. When $2m > n$, we note that there are no regular elliptic elements in $\mathfrak{q}$ (in fact, $\mathfrak{q}$ has no regular elements at all in this case). In this case, one might like to deduce that $L^2(\mathrm{Sp}(2n, \mathbb{R})/\,\mathrm{Sp}(2m, \mathbb{R}) \times \mathrm{Sp}(2n-2m, \mathbb{Z}))$ has no discrete series of

$\mathrm{Sp}(2n,\mathbb{R})$ occurring in its Plancherel formula. Unfortunately, our results are not that powerful. Instead, let $\mathfrak{t} \subset \mathrm{sp}(2n,\mathbb{R})$ be a compact Cartan subalgebra of $\mathfrak{g}$. Applying Corollary 6.3, we obtain

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma\in\operatorname{supp} L^2(\mathrm{Sp}(2n,\mathbb{R})/\,\mathrm{Sp}(2m,\mathbb{R})\times\mathrm{Sp}(2n-2m,\mathbb{Z}))\\ \sigma\in\widehat{\mathrm{Sp}(2n,\mathbb{R})}'_{\mathrm{temp}}}} \lambda_{\sigma,\mathfrak{t}}\right)\cap(i\mathfrak{t}^*)'=\{0\}$$

whenever $2m > n$. Again, we expect that $L^2(\mathrm{Sp}(2n,\mathbb{R})/\,\mathrm{Sp}(2m,\mathbb{R}) \times \mathrm{Sp}(2n - 2m,\mathbb{Z}))$ has no discrete series representations when $2m > n$, but the above statement does not yield that result. The above statement would still be true if there were finitely many discrete series in the above Plancherel formula or even if there were infinitely many discrete series in the Plancherel formula but their parameters were "bunched up near the singular set" in a certain way.

Example 6.6. Let us consider another family of examples. Let G_1 be a real, reductive algebraic group, and let $G = G_1 \times G_1 \times \cdots \times G_1$ be the n fold product of such groups. Consider the diagonal map

$$\Delta\colon G_1 \hookrightarrow G = G_1 \times G_1 \times \cdots \times G_1$$

by

$$\Delta(g) = (g, g, \ldots, g).$$

Then we consider the quotient $X = G/\Delta(G_1)$. We call such a space an *n-tuple* space. The case $n = 3$ are called *triple spaces*. Triple spaces were introduced and their geometry was studied for certain G_1 by Danielsen, Krötz, and Schlichtkrull [DKS]. The authors would like to thank Bernhard Krötz for suggesting that we consider this example. Note that we have

$$\widehat{G} = \{\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n |\ \sigma_j \in \widehat{G_1}\ \forall j\}$$

and

$$\widehat{G}_{\mathrm{temp}} = \{\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n |\ \sigma_j \in (\widehat{G_1})_{\mathrm{temp}}\ \forall j\}.$$

That is, the irreducible, unitary (resp. irreducible, tempered) representations of G are the tensor products of irreducible, unitary (resp. irreducible, tempered) representations of G_1. One checks from the criterion in Theorem 4.1 of [BK] that

$$\operatorname{supp} L^2(G/\Delta(G_1)) \subset \widehat{G}_{\mathrm{temp}}.$$

The case $n = 2$ is the group case. By this, we mean that

$$(G_1 \times G_1)/\Delta(G_1) \longrightarrow G_1$$

by

$$(g, h) \mapsto gh^{-1}$$

is a $G = G_1 \times G_1$ equivariant isomorphism of homogeneous spaces for $G = G_1 \times G_1$. In particular, for $n = 2$, the Plancherel formula for $L^2(G/\Delta(G_1))$ was proved by Harish-Chandra [HC76]. He showed that we have a decomposition

$$L^2(G/\Delta(G_1)) \cong \int_{\sigma\in(\widehat{G}_1)_{\mathrm{temp}}}^{\oplus} \sigma \otimes \sigma^* d\sigma$$

where $d\sigma$ is the Plancherel measure, supported on all of $(\widehat{G_1})_{\mathrm{temp}}$. In particular, as a representation of $G = G_1 \times G_1$, the support of the Plancherel measure is the antidiagonal inside $(\widehat{G_1})_{\mathrm{temp}} \times (\widehat{G_1})_{\mathrm{temp}}$. Philosophically, this makes sense. We quotiented out by the diagonal; the antidiagonal is what is left. Let us see what we can deduce from Corollary 6.1 and Corollary 6.3 in this case. One observes

$$i(\mathfrak{g}/\Delta(\mathfrak{g}_1))^* = \{(\xi_1, \xi_2) \in i\mathfrak{g}_1^* \times i\mathfrak{g}_1^* | \ \xi_1 + \xi_2 = 0\}$$

and the right hand side of the third statement of Corollary 6.1 becomes

$$\overline{\bigcup_{x \in X} i(\mathfrak{g}/\mathfrak{g}_x)^*} = \overline{\bigcup_{g \in G} \mathrm{Ad}^*(g) \cdot i(\mathfrak{g}/\Delta(\mathfrak{g}_1))^*} = \overline{\bigcup_{\substack{\mathcal{O} \subset i\mathfrak{g}^* \\ \mathcal{O} \text{ a } G \text{ orbit}}} (\mathcal{O} \times -\mathcal{O})}.$$

If $\sigma \in (\widehat{G_1})_{\mathrm{temp}}$ corresponds to the finite union of coadjoint orbits $\mathcal{O}_\sigma$, then the finite union of coadjoint orbits $\mathcal{O}_{\sigma^*}$ is the same as $-\mathcal{O}_\sigma$. Thus, the right hand side of (6.1) in Corollary 6.1 perfectly models the Plancherel formula. However, the left hand side involves an asymptotic cone; hence, what we can deduce from Corollary 6.1 is more limited.

To get a feel for what one can deduce from Corollary 6.1, let us specialize to the case $G_1 = \mathrm{SL}(2, \mathbb{R})$. Let us write σ_n^+ (resp. σ_n^-) for the holomorphic (resp. antiholomorphic) discrete series with parameter $n = 1, 2, \ldots$. Let us write $\sigma_{\nu,+}$ (resp. $\sigma_{\nu,-}$) for the spherical (resp. non-spherical) unitary principal series with parameter $\nu \in \mathbb{R}_{\geq 0}$. From Corollary 6.1 and Corollary 6.3, we can deduce

- There are at most finitely many pairs $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ for which
$$\mathrm{Hom}(\sigma_{m_1}^+ \otimes \sigma_{m_2}^+, L^2(G/\Delta(G_1))) \neq \{0\}$$
(The analogous statement holds for $\sigma_{m_1}^- \otimes \sigma_{m_2}^-$).
- For every $\epsilon > 0$, there are at most finitely many pairs $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ for which
$$\mathrm{Hom}(\sigma_{m_1}^+ \otimes \sigma_{m_2}^-, L^2(G/\Delta(G_1))) \neq \{0\}$$
and
$$|m_1 - m_2| \geq \epsilon \min\{m_1, m_2\}.$$
(The analogous statement holds for $\sigma_{m_1}^- \otimes \sigma_{m_2}^+$).
- For every $\epsilon > 0$, there are infinitely many pairs $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$ for which
$$\mathrm{Hom}(\sigma_{m_1}^+ \otimes \sigma_{m_2}^-, L^2(G/\Delta(G_1))) \neq \{0\}$$
and
$$|m_1 - m_2| < \epsilon \min\{m_1, m_2\}.$$
(The analogous statement holds for $\sigma_{m_1}^- \otimes \sigma_{m_2}^+$).
- The collection of $(m, \nu) \in \mathbb{N} \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2$ such that
$$\sigma_m^\pm \otimes \sigma_{\nu,\pm} \in \mathrm{supp}\, L^2(G/\Delta(G_1))$$
is bounded in $\mathbb{R}^2$. (The analogous statement holds for $\sigma_{\nu,\pm} \otimes \sigma_m^\pm$).
- For every $\epsilon > 0$, the collection of $(\nu_1, \nu_2) \in \mathbb{R}_{\geq 0}^2 \subset \mathbb{R}^2$ for which
$$\sigma_{\nu_1,\pm} \otimes \sigma_{\nu_2,\pm} \in \mathrm{supp}\, L^2(G/\Delta(G_1))$$
and
$$|\nu_1 - \nu_2| \geq \epsilon \min\{\nu_1, \nu_2\}$$
is bounded in $\mathbb{R}^2$.

- For every $\epsilon < 0$, the collection of $(\nu_1, \nu_2) \in \mathbb{R}^2_{\geq 0} \subset \mathbb{R}^2$ for which
$$\sigma_{\nu_1,\pm} \otimes \sigma_{\nu_2,\pm} \in \operatorname{supp} L^2(G/\Delta(G_1))$$
and
$$|\nu_1 - \nu_2| < \epsilon \min\{\nu_1, \nu_2\}$$
is unbounded in $\mathbb{R}^2$.

The reader might think the authors foolish for writing down weak statements regarding $\operatorname{supp} L^2(\mathrm{SL}(2,\mathbb{R}))$ when the Plancherel formula for this group was worked out more than 60 years ago [HC52]. We include these statements because they give a flavor of the sort of conclusions that one can draw from our asymptotic statements.

More generally, for arbitrary n and arbitrary G_1, one has
$$i(\mathfrak{g}/\Delta(\mathfrak{g}_1))^* = \{(\xi_1, \ldots, \xi_n) \in i\mathfrak{g}_1^* \times \cdots \times i\mathfrak{g}_1^* \Big| \sum_j \xi_j = 0\}.$$
The right hand side of (6.3) then becomes
$$\overline{\bigcup_{x \in X} i(\mathfrak{g}/\mathfrak{g}_x)^*} = \overline{\bigcup_{g \in G} \mathrm{Ad}^*(g) \cdot i(\mathfrak{g}/\Delta(\mathfrak{g}_1))^*}$$
$$= \overline{\{(\mathrm{Ad}^*(g_1)\xi_1, \ldots, \mathrm{Ad}^*(g_n)\xi_n) |\ \xi_j \in i\mathfrak{g}_1^*, \sum_j \xi_j = 0,\ g_j \in G_1\}}.$$
However, writing this set down explicitly for n greater than two is quite complicated in general. To illustrate this, consider the case $G_1 = \mathrm{SL}(2,\mathbb{R})$ and $n = 3$. Let
$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \Big| \ \theta \in \mathbb{R} \right\}$$
and
$$\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \Big| \ a \in \mathbb{R} \right\}$$
be the two standard representatives of conjugacy classes of Cartan subalgebras in $\mathfrak{g}_1 = \mathrm{sl}(2,\mathbb{R})$. Then there are eight conjugacy classes of Cartan subalgebras in $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_1 \times \mathfrak{g}_1$ with representatives $\mathfrak{t} \oplus \mathfrak{t} \oplus \mathfrak{t}$, $\mathfrak{t} \oplus \mathfrak{t} \oplus \mathfrak{a}$, $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{t}$, $\mathfrak{a} \oplus \mathfrak{t} \oplus \mathfrak{t}$, $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{a}$, $\mathfrak{a} \oplus \mathfrak{t} \oplus \mathfrak{a}$, $\mathfrak{a} \oplus \mathfrak{a} \oplus \mathfrak{t}$, and $\mathfrak{a} \oplus \mathfrak{a} \oplus \mathfrak{a}$. For each Cartan subalgebra $\mathfrak{b} \subset \mathfrak{g}$, we wish to find
$$i(\mathfrak{b}^*)'\text{-}\operatorname{supp} L^2(G/\Delta(G_1)) := \mathrm{AC}\left(\bigcup_{\substack{\sigma \in \widehat{G}'_{\mathrm{temp}} \\ \sigma \in \operatorname{supp} L^2(G/\Delta(G_1))}} \lambda_{\sigma,\mathfrak{b}} \right) \cap (i\mathfrak{b}^*)'.$$
One might call this set the regular, semisimple asymptotics of $\operatorname{supp} L^2(G/\Delta(G_1))$ in $i\mathfrak{b}^*$. By symmetry, we need only write this down for the cases $\mathfrak{b}_0 = \mathfrak{t} \oplus \mathfrak{t} \oplus \mathfrak{t}$, $\mathfrak{b}_1 = \mathfrak{t} \oplus \mathfrak{t} \oplus \mathfrak{a}$, $\mathfrak{b}_2 = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{a}$, and $\mathfrak{b}_3 = \mathfrak{a} \oplus \mathfrak{a} \oplus \mathfrak{a}$. One can check
$$i(\mathfrak{b}_3^*)'\text{-}\operatorname{supp} L^2(G/\Delta(G_1)) = i(\mathfrak{b}_3^*)',\ i(\mathfrak{b}_2^*)'\text{-}\operatorname{supp} L^2(G/\Delta(G_1)) = i(\mathfrak{b}_2^*)'.$$
In particular, the collection of
$$\sigma_{\nu_1,\pm} \otimes \sigma_{\nu_2,\pm} \otimes \sigma_{\nu_3,\pm} \in \operatorname{supp} L^2(G/\Delta(G_1))$$

is unbounded in the parameter $(\nu_1,\nu_2,\nu_3)\in\mathbb{R}^3_{>0}$ and it is unbounded in *every direction* in $\mathbb{R}^3_{>0}$. Similarly, the collection of

$$\sigma_m^{\pm}\otimes\sigma_{\nu_1,\pm}\otimes\sigma_{\nu_2,\pm}\in\operatorname{supp}L^2(G/\Delta(G_1))$$

is unbounded in the parameter $(m,\nu_1,\nu_2)\in\mathbb{R}^3_{>0}$ and it is unbounded in *every direction* in $\mathbb{R}^3_{>0}$. However, this is no longer true when we move to the Cartan subalgebra $\mathfrak{b}_1$. Break up

$$i\mathfrak{t}^*=i\mathfrak{t}^*_+\cup\{0\}\cup i\mathfrak{t}^*_-$$

such that for $G_1=\mathrm{SL}(2,\mathbb{R})$, $\lambda_{\sigma_m^+,\mathfrak{t}}\in i\mathfrak{t}^*_+$ for all m and $\lambda_{\sigma_m^-,\mathfrak{t}}\in i\mathfrak{t}^*_-$ for all m. Then

$$i(\mathfrak{b}_1^*)'\text{-}\operatorname{supp}L^2(G/\Delta(G_1))=[i\mathfrak{t}^*_+\times i\mathfrak{t}^*_-\times(i\mathfrak{a}^*)']\cup[i\mathfrak{t}^*_-\times i\mathfrak{t}^*_+\times(i\mathfrak{a}^*)'].$$

In particular, the collection

$$\sigma_n^+\otimes\sigma_m^-\otimes\sigma_{\nu,\pm}\in\operatorname{supp}L^2(G/\Delta(G_1))$$

is unbounded in the parameter $(n,m,\nu)\in\mathbb{R}^3_{>0}$. Further, if we take any open cone $\Gamma\subset\overline{\Gamma}\subset\mathbb{R}^3_{>0}$ whose closure is contained in $\mathbb{R}^3_{>0}$, then the collection of parameters $(n,m,\nu)\in\Gamma$ for which

$$\sigma_n^+\otimes\sigma_m^-\otimes\sigma_{\nu,\pm}\in\operatorname{supp}L^2(G/\Delta(G_1))$$

is unbounded. Analogous remarks hold for representations of type $\sigma_n^-\otimes\sigma_m^+\otimes\sigma_{\nu,\pm}$. On the other hand, if we continue to consider an open cone $\Gamma\subset\overline{\Gamma}\subset\mathbb{R}^3_{>0}$ whose closure is contained in $\mathbb{R}^3_{>0}$, then the collection of parameters $(n,m,\nu)\in\Gamma$ for which

$$\sigma_n^+\otimes\sigma_m^+\otimes\sigma_{\nu,\pm}\in\operatorname{supp}L^2(G/\Delta(G_1))$$

is bounded. This does not, however, imply that there are no representations of this form in $\operatorname{supp}L^2(G/\Delta(G_1))$, and it does not even imply that the collection of parameters (n,m,ν) for which the corresponding representation is in $\operatorname{supp}L^2(G/\Delta(G_1))$ is bounded in $\mathbb{R}^3_{>0}$ (being bounded in all cones whose closure is contained in this set is a weaker statement).

Finally, we move on to the fundamental Cartan subalgebra $\mathfrak{b}_0$. One has

$$\begin{aligned}&i(\mathfrak{b}_0^*)'\text{-}\operatorname{supp}L^2(G/\Delta(G_1))\\&=\{(\xi_1,\xi_2,\xi_3)\in i\mathfrak{t}^*\oplus i\mathfrak{t}^*\oplus i\mathfrak{t}^*|\ \xi_j\in i\mathfrak{t}^*_+,\ \xi_k,\xi_l\in i\mathfrak{t}^*_-,\ \xi_j+\xi_k+\xi_l\in i\mathfrak{t}^*_+\}\\&\bigcup\{(\xi_1,\xi_2,\xi_3)\in i\mathfrak{t}^*\oplus i\mathfrak{t}^*\oplus i\mathfrak{t}^*|\ \xi_j\in i\mathfrak{t}^*_-,\ \xi_k,\xi_l\in i\mathfrak{t}^*_+,\ \xi_j+\xi_k+\xi_l\in i\mathfrak{t}^*_-\}.\end{aligned}$$

If one breaks up $i(\mathfrak{b}_0^*)'$ into its eight octants, then one sees that two of these octants have empty intersection with the above set and the other six are "half" covered by the above set. One can deduce from this that there exists $m\in\mathbb{N}$ such that whenever

$$\operatorname{Hom}_G(\sigma_{m_1}^+\otimes\sigma_{m_2}^+\otimes\sigma_{m_3}^+,L^2(G/\Delta(G_1)))\neq\{0\}$$

we have $m_j,m_k\le m$ for distinct $j\neq k$. That is, at most one of the parameters m_1,m_2,m_3 can go off to ∞ at a time. Analogous results hold for the representations $\sigma_{m_1}^-\otimes\sigma_{m_2}^-\otimes\sigma_{m_3}^-$. One can also show that there exist infinitely many triples $(m_1,m_2,m_3)\in\mathbb{N}^3$ for which

$$\operatorname{Hom}_G(\sigma_{m_1}^+\otimes\sigma_{m_2}^+\otimes\sigma_{m_3}^-,L^2(G/\Delta(G_1)))\neq\{0\}.$$

Further, the collection of such triples heads off to ∞ in the direction (m_1,m_2,m_3) whenever $m_3\ge m_1+m_2$. Analogous results hold when the $+$ and $-$ signs are permuted or interchanged.

One can deduce analogous results for general $n > 3$ and $G_1 = \mathrm{SL}(2,\mathbb{R})$. We leave it to the reader to write them down. It might be an interesting exercise to write down some of these results for G_1 a higher rank group than $\mathrm{SL}(2,\mathbb{R})$.

Example 6.7. Finally, we give an example of Corollary 6.2. Suppose G is a real, reductive algebraic group, $H \subset G$ is a real, reductive algebraic subgroup, and $P \subset H$ is a parabolic subgroup of H. Let $P = MAN$ be the Langlands decomposition of P, let $(\chi, \mathbb{C}_\chi)$ be a unitary character of MA extended trivially on N to a character of P, and let

$$\mathcal{L}_\chi := G \times_P \mathbb{C}_\chi$$

be the corresponding G equivariant, Hermitian vector bundle on G/P. Then by Corollary 6.2, the set

$$\mathrm{AC}\left(\bigcup_{\substack{\sigma \in L^2(G/P, \mathcal{D}^{1/2}\otimes\mathcal{L}_\chi) \\ \sigma\in \widehat{G}'_{\mathrm{temp}}}} \mathcal{O}_\sigma\right) \cap (i\mathfrak{g}^*)'$$

is independent of the unitary character χ of P. Consider the case where $G_1 = \mathrm{SL}(2,\mathbb{R})$, $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $B \subset G_1 = \mathrm{SL}(2,\mathbb{R})$ is a Borel subgroup embedded diagonally in G. Then

$$\overline{\mathrm{Ad}^*(G)\cdot i(\mathfrak{g}/\Delta(\mathfrak{b}))^*} = \overline{\{(\xi_1,\xi_2) \in \mathfrak{g}_1^{\oplus 2} |\ \mathrm{Ad}^*(g_1)\xi_1 + \mathrm{Ad}^*(g_2)\xi_2 \in \mathcal{N} \text{ some } g_1, g_2 \in G_1\}}$$

where $\mathcal{N} \subset \mathrm{sl}(2,\mathbb{R})$ denotes the nilpotent cone. Analyzing this set together with Corollary 6.2 and Corollary 6.4, one arrives at several conclusions regarding

$$L^2(G/\Delta(B), \mathcal{D}^{1/2}\otimes\mathcal{L}_\chi),$$

all of which are independent of the character χ of B:

- There exists a natural number m such that whenever $(m_1, m_2) \in \mathbb{N}\times\mathbb{N}$ and
$$\mathrm{Hom}(\sigma^+_{m_1}\otimes\sigma^+_{m_2}, L^2(G/\Delta(B), \mathcal{D}^{1/2}\otimes\mathcal{L}_\chi)) \neq \{0\}$$
we must have $m_j \leq m$ for some j. (The analogous statement holds for $\sigma^-_{m_1}\otimes\sigma^-_{m_2}$).
- There are infinitely many pairs $(m_1, m_2) \in \mathbb{N}\times\mathbb{N}$ for which
$$\mathrm{Hom}(\sigma^+_{m_1}\otimes\sigma^-_{m_2}, L^2(G/\Delta(B), \mathcal{D}^{1/2}\otimes\mathcal{L}_\chi))) \neq \{0\}.$$
In fact, for any open cone $\Gamma \subset \overline{\Gamma} \subset \mathbb{R}^2_{>0}$, there are infinitely many such pairs $(m_1, m_2) \in \Gamma$. (The analogous statement holds for $\sigma^-_{m_1}\otimes\sigma^+_{m_2}$).
- The collection of $(m,\nu) \in \mathbb{N}\times\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$ such that
$$\sigma^\pm_m \otimes \sigma_{\nu,\pm} \in \operatorname{supp} L^2(G/\Delta(B), \mathcal{D}^{1/2}\otimes\mathcal{L}_\chi)$$
is unbounded in every direction in $\mathbb{R}^2_{\geq 0}$. (The analogous statement holds for $\sigma_{\nu,\pm}\otimes\sigma^\pm_m$).
- For every $\epsilon > 0$, the collection of $(\nu_1,\nu_2) \in \mathbb{R}^2_{\geq 0} \subset \mathbb{R}^2$ for which
$$\sigma_{\nu_1,\pm}\otimes\sigma_{\nu_2,\pm} \in \operatorname{supp} L^2(G/\Delta(B), \mathcal{D}^{1/2}\otimes\mathcal{L}_\chi)$$
is unbounded in every direction in $\mathbb{R}^2_{\geq 0}$.

In particular, we see that

$$\operatorname{supp} L^2([\mathrm{SL}(2,\mathbb{R})\times \mathrm{SL}(2,\mathbb{R})]/\Delta(B), \mathcal{D}^{1/2}\otimes \mathcal{L}_\chi)$$

is much larger than

$$\operatorname{supp} L^2([\mathrm{SL}(2,\mathbb{R})\times \mathrm{SL}(2,\mathbb{R})]/\Delta(\mathrm{SL}(2,\mathbb{R})))$$

for every unitary character χ of B.

Example 6.8. We end the section with a family of examples related to an interesting example of Kobayashi. In Theorem 6.2 of [Kob98c], Kobayashi considers the group $G=\mathrm{O}(p,q)$ and the subgroup $H=U(r,s)$ with $2r=p$ and $2s\leq q$. He assumes that p is positive and divisible by four. Under these assumptions, Kobayashi shows that there exist infinitely many distinct discrete series σ of $G=\mathrm{O}(p,q)$ for which

$$\mathrm{Hom}_G(\sigma, L^2(G/H))\neq\{0\}.$$

We note that the techniques of Kobayashi in [Kob98c] are very different from our own techniques. In particular, he utilizes his work on the theory of discretely decomposable restrictions ([Kob94], [Kob98a], [Kob98b]) together with his work on the decay of functions on certain types of homogeneous spaces [Kob97] and the classification of the discrete spectrum of reductive symmetric spaces ([FJ80], [MO84]; see also the exposition [Vog88]).

Since our work utilizes very different ideas, it is worth considering what we can show. Let $G=\mathrm{SO}(p,q)$ and $H=U(r,s)$ with $2r\leq p$ and $2s\leq q$ (we do not assume $2r=p$). In addition, assume that at least one of p and q is even and assume $p,q\geq 3$ (we do not assume that p or q is divisible by four). Further, let $\mathcal{L}\to G/H$ be any (possibly trivial) G-equivariant, Hermitian line bundle on G/H. It follows from Corollary 6.2 that there exist infinitely many distinct discrete series σ of $G=\mathrm{SO}(p,q)$ for which

$$\mathrm{Hom}_G(\sigma, L^2(G/H,\mathcal{L}))\neq\{0\}.$$

It is worth noting that even though our results are sometimes (though certainly not always) more general, the construction of discrete spectrum of Kobayashi [Kob98c] is more explicit. Therefore, there are advantages to both methods.

We can also give (very weak) negative results in related instances. For instance, let $G=\mathrm{SO}(p,q)$ and let $H=\mathrm{SO}(l,m)\times\mathrm{U}(r,s)$ with $l+2r\leq p$, $m+2s\leq q$, $2l>p$ and $2m>q$. Assume that at least one of p and q is even, and let $\mathfrak{t}\subset\mathfrak{g}=\mathfrak{so}(p,q)$ be a fundamental Cartan subalgebra. If $(i\mathfrak{t})^*$ denotes the set of regular, semisimple elements in $i\mathfrak{t}^*$ and $\Gamma\subset(i\mathfrak{t}^*)'$ is an open cone with $\overline{\Gamma}\subset(i\mathfrak{t}^*)'$, then there exists finitely many discrete series representations σ of $G=\mathrm{SO}(p,q)$ whose Harish-Chandra parameters meet Γ and for which

$$\mathrm{Hom}_G(\sigma, L^2(G/H))\neq\{0\}.$$

This follows from Corollary 6.3. Again one notes that there are likely no such discrete series σ, but our results do not imply that stronger statement.

7. Counterexamples and Whittaker Functionals

Let G be a real, reductive algebraic group, let $N\subset G$ be a unipotent subgroup, and let $(\chi,\mathbb{C}_\chi)$ denote a unitary character of N. If (σ, W_σ) is a unitary representation of G with smooth vectors W_σ^∞, then a *distribution Whittaker functional* on

(σ, W_σ) with respect to (N, χ) is a continuous, N-equivariant homomorphism

$$\psi\colon W_\sigma^\infty \longrightarrow \mathbb{C}_\chi.$$

We denote the vector space of such homomorphisms by $\mathrm{Wh}_{N,\chi}(\sigma)$. Distribution Whittaker functionals have primarily been studied in the special case where N is the nilradical of a parabolic subgroup $P \subset G$. In the case where (σ, W_σ) is an irreducible, tempered representation of G, the study of distribution Whittaker functionals was related to harmonic analysis by Harish-Chandra (unpublished) and independently by Wallach [Wal92]. Let us write down this relationship.

We may associate a Hermitian line bundle

$$\mathcal{L}_\chi = G \times_N \mathbb{C}_\chi \longrightarrow G/N$$

to the unitary character χ of N, and we may consider the unitary representation $L^2(G/N, \mathcal{L}_\chi)$ of G. First, by the Lemma on page 365 of [Wal92], we have

$$\operatorname{supp} L^2(G/N, \mathcal{L}_\chi) \subset \widehat{G}_{\mathrm{temp}}.$$

That is, the decomposition of $L^2(G/N, \mathcal{L}_\chi)$ into irreducibles consists entirely of irreducible, tempered representations. Next, the multiplicity of an irreducible, tempered representation σ in $L^2(G/N, \mathcal{L}_\chi)$ is equal to the dimension of the space of distribution Whittaker functionals for σ with respect to the pair (N, χ) (see the Theorem on page 425 of [Wal92]).

Now, the space $\mathrm{Wh}_{N,\chi}(\sigma)$ is not completely understood in general. However, some partial results exist in special cases ([Kos78], [Mat92], [GS]). For instance, consider the case where N is the nilradical of a minimal parabolic subgroup $P = MAN$. Note that $d\chi$ is trivial on the commutator algebra $[\mathfrak{n}, \mathfrak{n}]$, and therefore it descends to a linear functional

$$d\chi \in i(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*.$$

We say χ is nondegenerate if $d\chi$ is contained in an open MA orbit in $i(\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$. Matumoto proved a nice result under these conditions [Mat92].

Theorem 7.1 (Matumoto)**.** *Let G be a real, reductive algebraic group, let P be a minimal parabolic subgroup with nilradical N and Langlands decomposition $P = MAN$, and let χ be a nondegerate character of N. Then there exists a distribution Whittaker functional for an irreducible, unitary representation σ with respect to (N, χ) if and only if*

$$d\chi \in \mathrm{WF}(\sigma).$$

Here we identify $d\chi \in i\mathfrak{n}^* \subset i\mathfrak{g}^*$ in the usual way.

Combining Matumoto's Theorem with the above result of Harish-Chandra (and independently Wallach), one can determine $\operatorname{supp} L^2(G/N, \mathcal{L}_\chi)$ whenever G is a real, reductive algebraic group, $N \subset G$ is the nilradical of a minimal parabolic subgroup, and χ is a nondegenerate character of N. Combining this information with Theorem 1.2 of [HHO] on wave front sets of arbitrary direct integrals of tempered representations, one checks that in many cases

$$\mathrm{WF}(\mathrm{Ind}_N^G \chi) \supsetneq \mathrm{Ind}_N^G \mathrm{WF}(\chi) = \overline{\bigcup_{g \in G} \mathrm{Ad}^*(g) \cdot i(\mathfrak{g}/\mathfrak{n})^*}.$$

Let us write down the simplest example. Let $G = \mathrm{SL}(2,\mathbb{R})$, and let

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}.$$

The unitary characters of N are parametrized by $i\mathbb{R}$; let us write

$$\chi_\lambda \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e^{\lambda x}$$

for $\lambda \in i\mathbb{R}$. Now, we may form the Hermitian line bundle

$$\mathcal{L}_\lambda = G \times_N \chi_\lambda$$

and the unitary representation

$$L^2(G/N, \mathcal{L}_\lambda)$$

of $G = \mathrm{SL}(2,\mathbb{R})$ for every $\lambda \in i\mathbb{R}$.

To consider wave front sets, let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ denote the Lie algebra of $G = \mathrm{SL}(2,\mathbb{R})$. Introduce coordinates on the Lie algebra

$$\mathfrak{g} = \left\{ X_{x,y,z} = \begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

Notice that the G orbits on $\mathfrak{g}$ are the hyperboloids

$$x^2 + y^2 - z^2 = c$$

for $c > 0$, half of this hyperboloid when $c < 0$, and one of three pieces of the cone when $c = 0$. Notice that when $x^2 + y^2 - z^2 > 0$, the matrix $X_{x,y,z}$ is diagonalizable with real eigenvalues; we call such an element of $\mathfrak{g}$ *hyperbolic* and we denote the set of hyperbolic elements in $\mathfrak{g}$ by $\mathfrak{g}_{\mathrm{hyp}}$. When $x^2 + y^2 - z^2 < 0$, the matrix $X_{x,y,z}$ is diagonalizable with purely imaginary eigenvalues; we call such an element of $\mathfrak{g}$ *elliptic* and we denote the set of elliptic elements in $\mathfrak{g}$ by $\mathfrak{g}_{\mathrm{ell}}$. The set of nonzero elliptic elements has two connected components which we denote by $\mathfrak{g}_{\mathrm{ell}}^{+}$ (the set of nonzero elliptic elements $X_{x,y,z}$ with $z > 0$) and $\mathfrak{g}_{\mathrm{ell}}^{-}$ (the set of nonzero elliptic elements $X_{x,y,z}$ with $z < 0$). When $x^2 + y^2 - z^2 = 0$, the matrix $X_{x,y,z}$ is nilpotent, and we call such an element of $\mathfrak{g}$ *nilpotent*. We denote by $\mathfrak{g}_{\ textnilp}$ the set of nilpotent elements in $\mathfrak{g}$.

We may identify $\mathfrak{g} \cong \mathfrak{g}^*$ via the trace form

$$X \mapsto (Y \mapsto \mathrm{Tr}(XY))\,.$$

This isomorphism is G equivariant (so it takes G orbits on $\mathfrak{g}$ to G orbits on $\mathfrak{g}^*$). After dividing by i, we obtain a G invariant identification of $\mathfrak{g}$ with $i\mathfrak{g}^*$. We denote by $i\mathfrak{g}_{\mathrm{hyp}}^*$ (resp. $i\mathfrak{g}_{\mathrm{ell}}^*$, $i(\mathfrak{g}_{\mathrm{ell}}^*)^+$, $i(\mathfrak{g}_{\mathrm{ell}}^*)^-$, $i\mathfrak{g}_{\mathrm{nilp}}^*$) the subset of $i\mathfrak{g}^*$ which corresponds under the above isomorphism to $\mathfrak{g}_{\mathrm{hyp}}$ (resp. $\mathfrak{g}_{\mathrm{ell}}$, $\mathfrak{g}_{\mathrm{ell}}^+$, $\mathfrak{g}_{\mathrm{ell}}^-$, $\mathfrak{g}_{\mathrm{nilp}}$).

By Theorem 1.1, we know

$$\mathrm{WF}(L^2(G/N)) = \overline{\mathrm{Ad}^*(G) \cdot i(\mathfrak{g}/\mathfrak{n})^*}.$$

After identifying $i\mathfrak{g}^* \cong \mathfrak{g}$ via the trace form and dividing by i, one notes that $i(\mathfrak{g}/\mathfrak{n})^*$ corresponds to

$$\mathfrak{b} = \left\{ \begin{pmatrix} a & x \\ 0 & -a \end{pmatrix} \;\middle|\; a, x \in \mathbb{R} \right\}.$$

Now, all of the elements in $\mathfrak{b}$ are either hyperbolic or nilpotent, and all hyperbolic or nilpotent elements in $\mathfrak{g}$ are conjugate to elements in $\mathfrak{b}$. Thus, if we break up

$$i\mathfrak{g}^* = i\mathfrak{g}^*_{\mathrm{hyp}} \cup i\mathfrak{g}^*_{\mathrm{nilp}} \cup i\mathfrak{g}^*_{\mathrm{ell}},$$

then

$$\mathrm{WF}(L^2(G/N)) = i\mathfrak{g}^*_{\mathrm{hyp}} \cup i\mathfrak{g}^*_{\mathrm{nilp}}.$$

Let us consider the representation theory side for a moment. As we stated before, all of the irreducible representations of $G = \mathrm{SL}(2,\mathbb{R})$ occurring in the direct integral decomposition of $L^2(G/N,\mathcal{L}_\lambda)$ also occur in $L^2(G)$, that is, they are tempered. The irreducible, tempered representations of G are as follows. There are the holomorphic discrete series σ_n^+ for $n \in \mathbb{N}$, the antiholomorphic discrete series σ_n^- for $n \in \mathbb{N}$, the spherical unitary principal series $\sigma_{\nu,+}$ for $\nu \in \mathbb{R}_{\geq 0}$, the non-spherical unitary principal series $\sigma_{\nu,-}$ for $\nu \in \mathbb{R}_{>0}$, and the two limits of discrete series σ_0^+ and σ_0^-. We have the direct integral decomposition

$$L^2(G/N) \cong \int_{\nu\in i\mathbb{R}_{\geq 0}}^{\oplus} \sigma_{\nu,+} d\nu \bigoplus \int_{\nu\in i\mathbb{R}_{>0}}^{\oplus} \sigma_{\nu,-} d\nu.$$

Now, one computes from Theorem 1.2 of [HHO] that

$$\mathrm{WF}\left(\int_{\nu\in i\mathbb{R}_{\geq 0}}^{\oplus} \sigma_{\nu,+} d\nu \bigoplus \int_{\nu\in i\mathbb{R}_{>0}}^{\oplus} \sigma_{\nu,-} d\nu\right) = \overline{i\mathfrak{g}^*_{\mathrm{hyp}}} = i\mathfrak{g}^*_{\mathrm{hyp}} \cup i\mathfrak{g}^*_{\mathrm{nilp}}.$$

Of course, we observe that computing the wave front set from the L^2 side and the representation theory side yield the same thing. Now, let us consider the more general case $L^2(G/N,\mathcal{L}_\lambda)$ for $\lambda \neq 0$. We cannot compute the wave front set from the L^2 side using Corollary 6.2 because $\mathfrak{n} = \mathrm{Lie}(N)$ does not contain a dense subset of semisimple elements (in fact all of the elements in $\mathfrak{n}$ are nilpotent). But, we can still look at the representation theory side. Break up

$$i\mathfrak{n}^* = i\mathfrak{n}^*_+ \cup \{0\} \cup i\mathfrak{n}^*_-$$

so that

$$i\mathfrak{n}^*_+ \subset \overline{i(\mathfrak{g}^*_{\mathrm{ell}})^+},\ i\mathfrak{n}^*_- \subset \overline{i(\mathfrak{g}^*_{\mathrm{ell}})^-}.$$

Utilizing the work of Matumoto [Mat92], Harish-Chandra (unpublished), and Wallach [Wal92] together with knowledge of the wave front sets of irreducible, tempered representations of $\mathrm{SL}(2,\mathbb{R})$ (this knowledge can be derived from work of Rossmann [Ros78], [Ros80], [Ros95] and Barbasch-Vogan [BV80]; see Section 8.1 of [HHO] where this example is worked out in detail), we have

$$L^2(G/N,\mathcal{L}_\lambda) \cong \int_{\nu\in\mathbb{R}_{\geq 0}} \sigma_{\nu,+} d\nu \oplus \int_{\nu\in\mathbb{R}_{>0}} \sigma_{\nu,-} d\nu \oplus \sum_{n\in\mathbb{N}} \sigma_n^+.$$

if $d\chi_\lambda \in i\mathfrak{n}^*_+$ (the fact that the multiplicities are one was shown by Kostant [Kos78]). The analogous formula holds if $d\chi_\lambda \in i\mathfrak{n}^*_-$ with $+$ and $-$ swapped.

Now, applying Theorem 1.2 of [HHO], we obtain

$$\mathrm{WF}(L^2(G/N,\mathcal{L}_\lambda)) = i\mathfrak{g}^*_{\mathrm{hyp}} \cup i(\mathfrak{g}^*_{\mathrm{ell}})^+ \cup i\mathfrak{g}^*_{\mathrm{nilp}}.$$

if $d\chi_\lambda \in i\mathfrak{n}^*_+$. Similarly, we obtain

$$\mathrm{WF}(L^2(G/N,\mathcal{L}_\lambda)) = i\mathfrak{g}^*_{\mathrm{hyp}} \cup i(\mathfrak{g}^*_{\mathrm{ell}})^- \cup i\mathfrak{g}^*_{\mathrm{nilp}}.$$

if $d\chi_\lambda \in i\mathfrak{n}^*_-$.

In particular,

$$\begin{aligned}\mathrm{WF}(\mathrm{Ind}_N^G \chi_\lambda) &= i\mathfrak{g}^*_{\mathrm{hyp}} \cup i(\mathfrak{g}^*_{\mathrm{ell}})^+ \cup i\mathfrak{g}^*_{\mathrm{nilp}} \supsetneq i\mathfrak{g}^*_{\mathrm{hyp}} \cup i\mathfrak{g}^*_{\mathrm{nilp}} \\ &= \overline{\mathrm{Ad}^*(G)\cdot i(\mathfrak{g}/\mathfrak{n})^*} = \mathrm{Ind}_N^G \mathrm{WF}(\chi_\lambda)\end{aligned}$$

when $d\chi_\lambda \in i\mathfrak{n}^*_+$ and

$$\begin{aligned}\mathrm{WF}(\mathrm{Ind}_N^G \chi_\lambda) &= i\mathfrak{g}^*_{\mathrm{hyp}} \cup i(\mathfrak{g}^*_{\mathrm{ell}})^- \cup i\mathfrak{g}^*_{\mathrm{nilp}} \supsetneq i\mathfrak{g}^*_{\mathrm{hyp}} \cup i\mathfrak{g}^*_{\mathrm{nilp}} \\ &= \overline{\mathrm{Ad}^*(G)\cdot i(\mathfrak{g}/\mathfrak{n})^*} = \mathrm{Ind}_N^G \mathrm{WF}(\chi_\lambda)\end{aligned}$$

when $d\chi_\lambda \in i\mathfrak{n}^*_-$.

Hence, we have a counterexample to the converse to Theorem 1.1 of [HHO], and we have demonstrated the necessity of at least some hypothesis in Theorem 1.2 that does not exist in Theorem 1.1.

Appendix A. Density bundles

Let X be an n-dimensional smooth manifold and let $\mathcal{F}X \to X$ be the frame bundle whose fibers over the base point $x \in X$ consist of all ordered bases of the n-dimensional vector space T_xX. Note that a choice of such a basis is equivalent to a choice of a linear isomorphism $b : \mathbb{R}^n \to T_xX$ and we will denotes points in $\mathcal{F}X$ by two-tuples (x, b). Given any $g \in \mathrm{GL}(n, \mathbb{R})$ we can define canonically its right action on $\mathcal{F}X$ by the pullback of the isomorphism $b : \mathbb{R}^n \to T_xX$ with g

$$(x, b)g := (x, b \circ g).$$

This action is free and transitive on the fibers, so $\mathcal{F}X$ is a principle $\mathrm{GL}(n, \mathbb{R})$ fiber bundle.

If $U \subset X$ and $V \subset \mathbb{R}^n$ are open and

$$\kappa : U \subset X \to V \subset \mathbb{R}^n$$

is a smooth chart, then this chart naturally gives rise to a local section of the frame bundle defined by

$$\tau_\kappa : U \to \pi^{-1}_{\mathcal{F}X}(U),\ x \mapsto (x, d\kappa^{-1}_{|\kappa(x)}). \tag{A.1}$$

For $\alpha > 0$ the map $\mathrm{GL}(n, \mathbb{R}) \to \mathrm{End}(\mathbb{C}), g \mapsto |\det(g)|^{-\alpha}$ is a one dimensional representation and we can define the density bundles as the associated fiber bundles with respect to these representations.

Definition A.1. For a smooth n-dimensional manifold X and $\alpha > 0$ we define the α-density bundle over X as

$$\mathcal{D}^\alpha := \mathcal{F}X \times_{|\det(\bullet)|^{-\alpha}} \mathbb{C}.$$

As $\mathbb{R}_{\geq 0} \subset \mathbb{C}$ is invariant under the action by $|\det(\bullet)|^{-\alpha}$ we can also define the positive α density bundle as

$$\mathcal{D}^\alpha_{\geq 0} := \mathcal{F}X \times_{|\det(\bullet)|^{-\alpha}} \mathbb{R}_{\geq 0}.$$

We will denote elements in the density bundles by equivalence classes $[(x, b), z] \in \mathcal{D}^\alpha$ where $(x, b) \in \mathcal{F}X$, $z \in \mathbb{C}$ and the equivalence relation is given by $((x, b\circ g), z) \sim ((x, b), |\det(g)|^{-\alpha} z)$.

Note that the density bundles behave nicely under tensor products as we have for $\alpha, \beta > 0$

$$\mathcal{D}^\alpha \otimes \mathcal{D}^\beta \cong \mathcal{D}^{\alpha+\beta}. \tag{A.2}$$

Moreover, there is a global absolute value map

$$| \quad |_{\mathcal{D}^\alpha} : \Gamma(\mathcal{D}^\alpha) \to \Gamma(D^\alpha_{\geq 0}) \tag{A.3}$$

given by

$$[(x,b),z] \mapsto [(x,b),|z|].$$

Sometimes it is useful to work in coordinates. Given a smooth chart $\kappa : U \to V$ of X and using the local sections τ_κ defined in (A.1) we obtain a local trivialization of the density bundles and can thus locally identify sections $\Psi : U \to \pi^{-1}_{\mathcal{D}^\alpha}(U)$ with a function $\Psi_\kappa : V \to \mathbb{C}$ by the condition

$$\Psi(m) = [\tau_\kappa(m), \Psi_\kappa(\kappa(m))]$$

which determines the function Ψ_κ uniquely as the right $\mathrm{GL}(n,\mathbb{R})$ action on the fiber is free.

If $\kappa' : U \to V'$ is another chart then

$$\begin{aligned}
\Psi(m) &= [\tau_\kappa(m), \Psi_\kappa(\kappa(m))] \\
&= [(m, d\kappa^{-1}_{|\kappa(m)}), \Psi_\kappa(\kappa(m))] \\
&= [(m, d\kappa'^{-1}_{|\kappa'(m)} \circ d(\kappa' \circ \kappa^{-1})_{|\kappa(m)}), \Psi_\kappa(\kappa(m))] \\
&= [\tau_{\kappa'}(m), |\det(d(\kappa' \circ \kappa^{-1})_{|\kappa(m)})|^{-\alpha} \cdot \Psi_\kappa(\kappa \circ \kappa'^{-1}(\kappa'(m))].
\end{aligned}$$

Consequently we have for $y \in V'$

$$\begin{aligned}
\Psi_{\kappa'}(y) &= |\det(d(\kappa' \circ \kappa^{-1})_{|\kappa\circ\kappa'^{-1}(y)})|^{-\alpha} \cdot \Psi_\kappa(\kappa \circ \kappa'^{-1}(y)) \\
&= |\det(d(\kappa \circ \kappa'^{-1})_{|y})|^{\alpha} \cdot \Psi_\kappa(\kappa \circ \kappa'^{-1}(y)).
\end{aligned} \tag{A.4}$$

Note that the same construction associates sections ρ of the positive density bundle $\mathcal{D}^\alpha_{\geq 0}$ to functions $\rho_\kappa : V \to \mathbb{R}_{\geq 0}$ and the transformation with respect to coordinate change is also according to (A.4).. We can thus give an alternate definition of the global absolute value map for sections on the density bundle $| \quad |_{\mathcal{D}^\alpha} : \Gamma(\mathcal{D}^\alpha) \to \Gamma(D^\alpha_{\geq 0})$ by requiring locally for a chart $\kappa : U \to V$ and $x \in V$ that

$$(|\Psi|_{\mathcal{D}^\alpha})_\kappa(x) := |\Psi_\kappa(x)|.$$

Note that (A.4) assures that this definition is chart independent. It is easy to see that this coordinate definition agrees with the definition (A.3).

Given a chart $\kappa : U \to V$ and a section $\Psi \in \Gamma(\mathcal{D}^1)$ compactly supported in U, we say that Ψ is *integrable* if and only if Ψ_κ is Lebesgue integrable on $V \subset \mathbb{R}^n$ and we define

$$\int_U \Psi := \int_V \Psi_\kappa(x) d\lambda(x) \tag{A.5}$$

where $d\lambda(x)$ is the usual Lebesgue measure. The behavior of Ψ_κ under coordinate changes (A.4) assures that this definition is independent of the choice of charts. The same definition holds for sections in the positive density bundle $\rho \in \Gamma(\mathcal{D}^1_{\geq 0})$. Note that from the definition of the global absolute value map (A.3) we directly obtain, that Ψ is integrable if and only if $|\Psi|_{\mathcal{D}^1}$ is integrable and that

$$\left| \int_U \Psi \right| \leq \int_U |\Psi|_{\mathcal{D}^1}. \tag{A.6}$$

An arbitrary not necessarily compactly supported section $\Psi \in \Gamma(\mathcal{D}^1)$ is said to be integrable if for a countable atlas (κ_i, U_i, V_i) and a partition of unity χ_i subordinate

to the cover U_i we have that for all i, $\chi_i\Psi$ are integrable as compactly supported sections and that

$$\sum_i \left(\int_{U_i} |\chi_i\Psi|_{\mathcal{D}^1}\right) < \infty.$$

We then define

$$\int_X \Psi := \sum_i \left(\int_{U_i} \chi_i\Psi\right).$$

Again one checks, that this definition is independent of the choice of the atlas and partition of unity using (A.4). As above the same definition applies for sections in the positive density bundle. Furthermore we have that an arbitrary section $\Psi \in \Gamma(\mathcal{D}^1)$ is integrable if and only if $|\Psi|_{\mathcal{D}^1}$ is integrable and the inequality (A.6) holds. If $\rho_1, \rho_2 \in \mathcal{D}^1_{\geq 0}$ then we say $\rho_1 \leq \rho_2$ if this equality holds fiber wise. We then immediately obtain from the definition of the integrals

$$\int_X \rho_1 \leq \int_X \rho_2. \tag{A.7}$$

Let $\mathcal{V} \to X$ be a Hermitian vector bundle, where the fibers $\mathcal{V}_x$ are Hilbert spaces with scalar products $\langle\ ,\ \rangle_{\mathcal{V}_x}$ respectively norms $\|\ \ \|_{\mathcal{V}_x}$. By tensoring with the density bundles we can define L^1 and L^2-norms of sections in $\mathcal{V} \otimes \mathcal{D}^\alpha$ ($\alpha = 1, 1/2$) as follows: Given a section

$$f \in \Gamma(\mathcal{V} \otimes \mathcal{D}^\alpha), f : x \to v_x \otimes z_x$$

we can associate a section $\|f\|_{\mathcal{V}\otimes\mathcal{D}^\alpha}$ in $\mathcal{D}^\alpha_{\geq 0}$ by setting

$$\|f\|_{\mathcal{V}\otimes\mathcal{D}^\alpha} : x \mapsto \|v_x\|_{\mathcal{V}_x} \cdot |z_x|_{\mathcal{D}^\alpha}.$$

We now say that the L^1-norm of $f \in \Gamma(\mathcal{V} \otimes \mathcal{D}^1)$ (respectively the L^2-norm of $f \in \Gamma(\mathcal{V} \otimes \mathcal{D}^{1/2})$) is defined if $\|f\|_{\mathcal{V}\otimes\mathcal{D}^1}$ (respectively $(\|f\|_{\mathcal{V}\otimes\mathcal{D}^{1/2}})^{\otimes 2}$) is integrable and set

$$\|f\|_{L^1} := \int_X \|f\|_{\mathcal{V}\otimes\mathcal{D}^1},$$

respectively

$$\|f\|_{L^2} := \sqrt{\int_X (\|f\|_{\mathcal{V}\otimes\mathcal{D}^{1/2}})^{\otimes 2}}.$$

We define for $p = 1, 2$

$$\mathcal{L}^p(\mathcal{V} \otimes \mathcal{D}^{1/p}) := \left\{ f \in \Gamma(\mathcal{V} \otimes \mathcal{D}^{1/p}), \text{s.t. } \|f\|_{L^p} \text{ is defined } \right\}$$

and

$$L^p(\mathcal{V} \otimes \mathcal{D}^{1/p}) := \mathcal{L}^p(\mathcal{V} \otimes \mathcal{D}^{1/p}) / \{f \in \mathcal{L}^p(\mathcal{V} \otimes \mathcal{D}^{1/p}),\ \|f\|_{L^p} = 0\}$$

then $L^1(\mathcal{V} \otimes \mathcal{D}^{1/p})$ becomes a Banach space with norm $\|\ \ \|_{L^1}$ and $L^2(\mathcal{V} \otimes \mathcal{D}^{1/2})$ a Hilbert space with the scalar product

$$\langle f_1, f_2 \rangle_{L^2} := \int \langle f_1(x), f_2(x) \rangle_{\mathcal{V}_x \otimes \mathcal{D}_x^{1/2}}.$$

Here $\langle f_1(x), f_2(x) \rangle_{\mathcal{V}_x \otimes \mathcal{D}_x^{1/2}}$ denotes the section in $\mathcal{D}^1$ which is assigned to the two sections $f_i : x \mapsto v_i(x) \otimes z_i(x)$, $i = 1, 2$ by

$$x \mapsto \langle v_1(x), v_2(x) \rangle_{\mathcal{V}_x} z_1(x) \otimes \overline{z_2}(x).$$

If the base manifold X is equipped with a smooth left action by a Lie group G, then this action can be lifted to $\mathcal{D}^\alpha$ as follows. Recall that for $(x, b) \in \mathcal{F}_x X$,

$b : \mathbb{R}^n \to T_xX$ is a linear isomorphism. Now for any $g \in G$ the differential of this left action is a linear isomorphism between the tangent spaces $dg_{|x} : T_xX \to T_{gx}X$. Thus we can define

$$l_g : \mathcal{F}_xX \to \mathcal{F}_{gx}X, \qquad (x,b) \mapsto (gx, dg_{|x} \circ b).$$

This left action on the frame bundle then leads to a canonical left action on the density bundles

$$l_g : \mathcal{D}_x^\alpha \to \mathcal{D}_{gx}^\alpha, \qquad [(x,b),z] \mapsto [(gx, dg_{|x} \circ b), z].$$

If $h \in G_x$ lies in the stabilizer of the point $x \in X$, then l_h is a linear isomorphism of the fiber $\mathcal{D}_x^\alpha \cong \mathbb{C}$ and we get a one dimensional representation $(\sigma_x, \mathcal{D}_x^\alpha)$ of G_x which is in general not unitary. We can even explicitly calculate this representation:

$$\begin{aligned} \sigma_x(h)[(x,b),z] &= [(x, dh_{|x} \circ b), z] \\ &= [(x, b \circ b^{-1} \circ dh_{|x} \circ b), z] \\ &= [(x,b), |\det_{\mathbb{R}^n}(b^{-1} \circ dh_{|x} \circ b)|^{-\alpha} z] \\ &= [(x,b), |\det_{T_xX}(dh_{|x})|^{-\alpha} z]. \end{aligned} \tag{A.8}$$

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Department of Mathematics, Oklahoma State University, Stillwater, OK 74078
E-mail address: `Benjamin.Harris@math.okstate.edu`

Institut für Mathematik, Universitat Paderborn, Paderborn, Germany
E-mail address: `weich@math.uni-paderborn.de`